Lattice Model Exams, Jan 2017.

(1) Express the expected number of visits of the vertex $(2,0,1,7)$ of a simple random walk on $\mathbb{Z}^4$ starting from $(0,0,0,0)$ as a quadruple integral.

(2) Write down a 6×6 matrix M such that $|\det M|$ equals the number of domino tilings of a 3×4 chessboard.

(3) Let $R=[-1,1]\times[0,1]$ be a rectangular box. For $\delta=\frac{1}{n}$, with $n\in\mathbb{N}$, let $R_\delta := R\cap\delta\mathbb{Z}^2$ and for $(x,y)\in R_\delta$ let $P_\delta(x,y)$ be the probability that a simple random walk starting from (x,y) hits the top side of R_δ before any of the three other sides (left, bottom, right). Show that $P_\delta(x,y)\leq y$.

(4) Let $R=[a,b]\times[c,d]$ be a rectangular box. Discretize R by a fine hexagonal lattice of mesh size $\delta>0$ and consider the usual critical percolation (color independently the faces in black/white with probability $\frac{1}{2}/\frac{1}{2}$). Let A be the probability that there is a cluster linking the left side to the right side of the box. Let B be the probability that there is a black cluster touching all four sides of the rectangular box. Show that $\mathbb{P}(B)\geq A(1-A)$.

(5) Let G be a finite connected graph with vertices labelled $1,2,\ldots,n$. Let N be the number of spanning trees of G. Let P be the probability that a loop-erased random walk from 1 to n passes through the vertices $1,2,3,\ldots,n-1,n$ in that order (assuming that it is possible to go through them in that order). Show that $P=1/N$. Hint: Wilson's algorithm.

(6) Show that a discrete harmonic function $f:\mathbb{Z}^2\to\mathbb{R}$ that is bounded is constant. Hint: use the discrete Harnack inequality.

(7) Using Wilson's algorithm, explain why for any finite connected graph G, the law of the edges visited by a loop-erased random walk from x to y is the same as the law of the edges visited by a loop-erased random walk from y to x. Hint: use the fact that Wilson's algorithm doesn't depend on the order in which we label the vertices of the graph.

(8) Consider a discretization $\mathbb{D}_\delta=\mathbb{D}\cap\delta\mathbb{Z}^2$, and consider the Ising model with + boundary conditions at inverse temperature $\beta>0$. Show that there exist $\beta>0$ such that $\liminf_{\delta\to 0}\mathbb{E}^{\beta}_{\mathbb{D}_\delta;+}\left[\sigma_{(0,0)}\right]\geq 0.99$. Harder: explain why there is a limit as $\delta\to 0$.

(9) Consider the Ising model. Show that for any connected graph G (no boundary conditions) and any inverse temperature $\beta>0$ and any vertices $x,y\in G$, we have $\mathbb{E}\left[\sigma_x\sigma_y\right]>0$. Hint: high-temperature expansion.

(10) Let Ω be a domain such that $\partial\Omega$ is a simple curve, with three points $a_1,a_2,a_3\in\partial\Omega$ appearing in counterclockwise order. Let Ω_δ be the discretization of Ω by a hexagonal lattice of mesh size $\delta>0$. Consider the usual critical percolation on the faces of Ω_δ. For $z\in\Omega$, let $H_1^\delta(z)$ be the probability that a_1,z are separated from a_2,a_3 by a black path, and let $H_2^\delta(z)$ and $H_3^\delta(z)$ denote the symmetrical events. We have seen in class that for any simple, smooth closed oriented curve γ, if we discretize it into an oriented closed path of edges of the hexagonal lattice γ_δ, we have (identifying points with their positions in the complex plane).

$$\lim_{\delta\to 0}\sum_{\vec{xy}\in\gamma_\delta}\frac{\left(H_1^\delta+H_2^\delta+H_3^\delta\right)(x)+\left(H_1^\delta+H_2^\delta+H_3^\delta\right)(y)}{2}(y-x)=0$$

From this, explain why $\lim_{\delta\to 0}\left(H_1^\delta+H_2^\delta+H_3^\delta\right)(z)=1$ for any $z\in\Omega$, using the following hints:
Morera $\Longrightarrow$ *Holomorphicity*
Cauchy-Riemann $\Longrightarrow$ *Something about purely real holomorphic functions*
RSW $\Longrightarrow$ *Boundary Conditions*

(11) Let $R=[a,b]\times[c,d]$ be a rectangular box. Discretize R by a fine hexagonal lattice of mesh size $\delta>0$, and consider the usual critical percolation on it. Let E_{wwbwbb} and E_{wbwbwb} be the events that going from left to right in the box, we can find six disjoint paths of colors white-white-black-white-black-black and white-black-white-black-white-black going from left to right respectively. Show that $\mathbb{P}(E_{wwbwbb})=\mathbb{P}(E_{wbwbwb})$. *Hint: color flipping.*