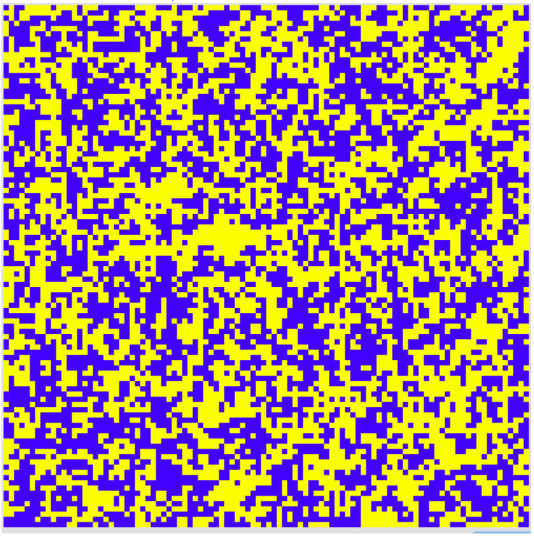
Examination

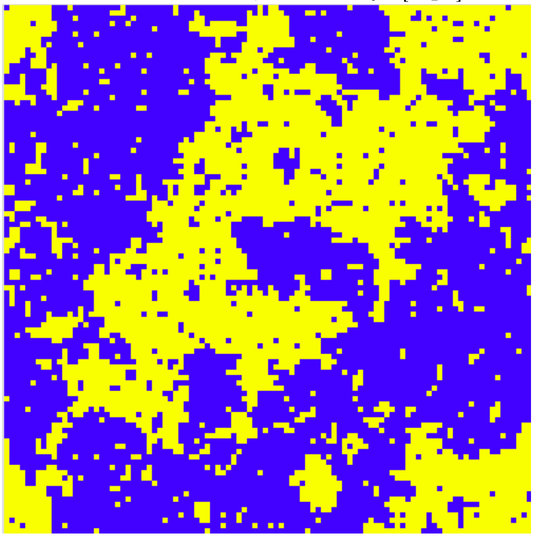
Lattice Models - EPFL 2018-2019

January 15, 2019

Justify all your questions. You do not have to prove again the things seen during the lesson.

1. We simulated two Ising models on a square grid domain for two different $\beta_1 < \beta_2$ (+1 spins in blue, −1 spins in yellow). Which picture is more likely to be the one we obtained for β_1? Justify. [3 pt]





2. Let us consider two simple random walks $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ such that $X_0 = 20$ and $Y_0 = 18$. Then $X_{\max(Y_n,0)}$ visits 2019 almost surely. True or False? Justify in either case! [1 + 6 pt]

3. Let $G \subset \mathbb{Z}^2$ be a bounded connected domain with $-$ boundary conditions. Consider the Ising model on G with parameter $\beta > 0$. There exist x and y in G such that $\mathbb{E}[\sigma_x]\,\mathbb{E}[\sigma_y] \leq 0$. True or False? Justify in either case! [1 + 6 pt]

4. Let $G \subset \mathbb{Z}^2$ be a bounded connected domain with + boundary conditions. Consider the low temperature expansion (L.T.E.) of the Ising model with parameter $\beta > 0$. For any distinct x and y in G, $\mathbb{E}[\sigma_x \sigma_y]$ is strictly smaller than:

$$\mathbb{P}(x \text{ and } y \text{ are not separated by any loop in the LTE}) - \mathbb{P}(x \text{ and } y \text{ are separated by at least one loop in the LTE}).$$

True or False? Justify in either case! [1 + 6 pt]

5. For $n \geq 1$, consider the $2n \times 2n$ chessboard C with black cells $\mathcal{B} = \{b_1, \ldots, b_{2n^2}\}$ and white cells $\mathcal{W} = \{w_1, \ldots, w_{2n^2}\}$. Let A denote the $2n^2 \times 2n^2$ reduced adjacency matrix ($A_{i,j} = 1$ if b_i adjacent to w_j, 0 if not). Then $|\det A| < \#\{\text{Domino tilings of } C\}$. True or False? Justify in either case! [1 + 6 pt]

6. Let us consider a discretization of a square $[0,1]\times[0,1]$ by a honeycomb lattice of small mesh size and color the hexagons of D_δ in black or white with probability $1/2$ in an independent fashion. Let $L=[0,1]\times\{0\}$ and $U=[0,1]\times\{1\}$. Then

$$\mathbb{P}\{\text{There exists a black and a white path both linking } L \text{ and } U\} \leq \frac{1}{4}.$$

True or False? Justify in either case! [1 + 6 pt]

7. Consider a random domino tiling of the 8×8 chessboard, picked uniformly at random. Show that the probability that the top row is occupied by 4 adjacent horizontal dominos is equal to

$$\sqrt{\frac{\prod_{k=1}^{8} \prod_{\ell=1}^{7} \left(2\cos\left(\frac{\pi k}{9}\right) + 2i\cos\left(\frac{\pi \ell}{8}\right)\right)}{\prod_{k=1}^{8} \prod_{\ell=1}^{8} \left(2\cos\left(\frac{\pi k}{9}\right) + 2i\cos\left(\frac{\pi \ell}{9}\right)\right)}}.$$

[6 pt]

8. For $N \geq 1$, consider the graph G_N consisting of the $3N-2$ vertices $\{n, ne^{2\pi i/3}, ne^{4\pi i/3} : n = 0, \ldots, N\}$ where two vertices are adjacent if they are at (Euclidean) distance 1 from each other. (Make a picture) For $n = 0, \ldots, N$, compute the probability that a simple random walk on G_N starting from n hits $Ne^{4\pi i/3}$ before $\{N, Ne^{2\pi i/3}\}$. [6 pt]

9. Consider the domain $D = \{z^2 : 0 \leq \Re\mathrm{e}\,(z)\,, \Im\mathrm{m}\,(z) \leq 1\}$. Discretize D_δ by a honeycomb lattice of mesh size δ and color the hexagons of D_δ in black or white with probability $1/2$ in an independent fashion. Let P_δ denote the probability that the points $\frac{1}{4}$ and $-\frac{1}{4}$ are separated by a black path from the points $\frac{3}{4} + i$ and $-\frac{3}{4} + i$ in D_δ. Show that $\lim_{\delta \to 0} P_\delta = \frac{1}{2}$. [6 pt]

10. Consider a modified simple random walk on $\mathbb{Z}$ starting from 0 which jumps with probability $\frac{3}{4}$ to the right nearest neighbor and with probability $\frac{1}{4}$ to the left nearest neighbor. Show that the expected number of visits of 0 is finite. Show that it is equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{1-\frac{1}{2}\cos{(\xi)}-\frac{1}{2}e^{i\xi}}d\xi.$$

[6 pt]

11. Suppose we want to apply Wilson's algorithm on $\mathbb{Z}^d$, by taking a root and v_0 labelling the vertices of $\mathbb{Z}^d = \{v_1, \ldots, v_n, \ldots\}$ arbitrarily (using a bijection between $\mathbb{N}$ and $\mathbb{Z}^d$), thus defining (if the algorithm works) a growing sequence of trees $\{v_0\} = T_0 \subset T_1 \subset \ldots$. Will the algorithm work and will any finite box be included in some T_k tree for large enough k ? Justify your answer. [6 pt]

12. (A bit harder) Let us consider a finite graph G with a fixed root v_0. Let $v_1, \ldots, v_n$ be a labelling of the vertices of G. Suppose we apply Wilson's algorithm following the labelling $v_1, \ldots, v_n$ and sample the LERW by sampling SRW and erasing their loops in chronological order. Let K denote the total number of steps of all these SRW. Show that the law of K does not depend on the labelling $v_1, \ldots, v_n$, but that it may depend on v_0. [6 + 4 pt]

13. (Harder) Find a function $f : \mathbb{Z}^3 \to \mathbb{R}$ which is discrete harmonic on $\mathbb{Z}^3$ except at $\{(0,0,0)\}$, bounded but not constant. Justify! [10 pt]

14. What was your favorite topic covered in class? [1 pt]