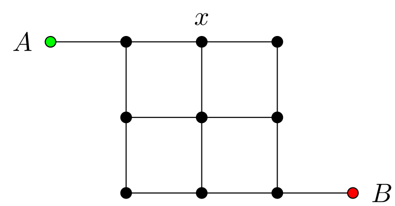
LATTICE MODELS

EXAMINATION
EPFL 2022-23

Justify all your answers. You do not have to reprove any result seen in the lectures.

Exercise 1.
In the following graph, compute the probability that a simple random walk starting from x hits A before B.



Exercise 2.
Consider the following random snake game $(X_n)_{n\geq 0}$ in $\mathbb{Z}^2$:

— The snake starts at $X_0 = (0,0)$.
— For every $n \geq 0$, X_{n+1} is chosen uniformly at random among the neighbors of X_n in $\mathbb{Z}^2 \setminus \{X_0, \ldots, X_{n-1}\}$.
— If there are no such neighbors, the snake dies.

Show that the snake dies eventually with probability 1.

Exercise 3.

Let $\Omega \subset \mathbb{C}$ be a smooth domain such that $e^{2\pi i/3}\Omega = \Omega$, i.e. Ω is invariant under a $\frac{2\pi}{3}$ rotation around the origin, and let $x \in \partial\Omega \setminus \{0\}$. For $\delta > 0$, let Ω_δ be the δ-meshed discretization of Ω by a honeycomb lattice, and let x_δ be a point of Ω_δ at distance $\leq \delta$ from x.

Consider critical percolation on the faces of Ω_δ, i.e. independent random colouring in black/white of the faces with probability 1/2. Compute, with justification,

$$\lim_{\delta \to 0} \mathbb{P}\left\{x_\delta \text{ and } 0 \text{ are separated by a black path from } e^{2\pi i/3}x_\delta \text{ and } e^{4\pi i/3}x_\delta\right\}.$$

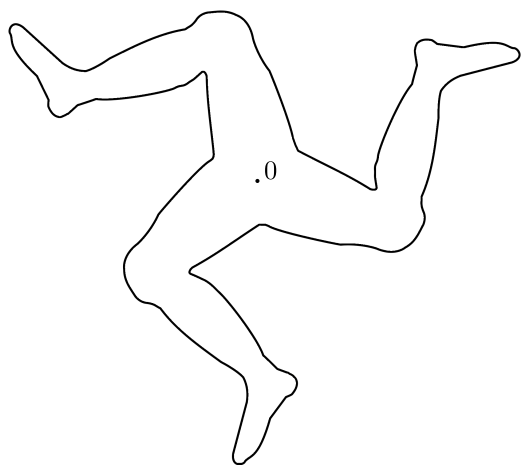


FIGURE 1. A domain with the same symmetry as Ω

Exercise 4.

Consider a graph $G = (V, E)$ with $V = B \cup W$ such that if $b, b' \in B$ then $b \not\sim b'$ and if $w, w' \in W$ then $w \not\sim w'$, with $|B| = |W| = n$. Let A be the $n \times n$ reduced adjacency matrix indexed by $B \times W$, i.e.

$$A_{bw} = \begin{cases} 1 & \text{if } b \sim w \\ 0 & \text{if } b \not\sim w. \end{cases}$$

Suppose that G has no dimer cover. Show that $\det A = 0$.

Exercise 5.
Consider X_n a simple random walk on $\mathbb{Z}^2$ starting from $(0,0)$ and $\tilde{X}_n$ a simple random walk on $\mathbb{Z}^2$ starting from $(1,1)$. Show that X_n and $\tilde{X}_n$ can be coupled so that with probability 1, there exists a (random) N such that $X_{N+k} = \tilde{X}_{N+k}$ for all $k \in \mathbb{N}$.

Exercise 6.
Let $G = (V, E)$ be a finite graph, let W be a subset of V, let v be in $V \setminus W$ and let w be in W. We consider a LERW $v = X_0, \ldots, X_T = w$ from v to w. Let τ be the stopping time such that

$$X_i \in V \setminus W \quad \text{for } 0 \leq i \leq \tau$$
$$X_{\tau+1} \in W.$$

Show that the law of $X_{\tau+1}, \ldots, X_T$ knowing $X_0, \ldots, X_\tau$ is that of a LERW in $V \setminus \{X_0, \ldots, X_{\tau-1}\}$ from X_τ to w.

Exercise 7.
In this exercise we consider a modified version of Wilson's algorithm on a graph $G = (V, E)$ using Antichronological Loop-Erased Random Walks (ACLERW).

For a path $(X_0, \ldots, X_n)$ in G, we define $\lambda^{\leftarrow}(X_0, \ldots, X_n)$ as the antichronological loop erasure of $(X_0, \ldots, X_n)$, obtained by removing the loops it has formed backwards in time, i.e. we perform the classical LERW erasure procedure on the path $X_n, X_{n-1}, \ldots, X_0$.

Question 1. Draw an example of a graph and a path where the antichronological loop erasure is different from the usual loop erasure.

For $W \subset V$ and $x \in V \backslash W$, an ACLERW from x to W is defined as $\lambda^{\leftarrow}(X_0, \ldots, X_\tau)$. where $(X_0, \ldots, X_\tau)$ is a a simple random walk on G starting at x and stopped upon hitting W (at time τ).

Question 2. Show that constructing a spanning tree by replacing the (classical) LERWs by ACLERWs in Wilson's algorithm also yields a uniform spanning tree.

Exercise 8.
Let I_n denote the discrete interval $[1,n]\cap\mathbb{Z}$ with boundary $\partial_n=\{0,n+1\}$ and let $\bar{I}_n=I_n\cup\partial_n$. Additionally, let $Q_n=I_n\times I_n$, $\partial Q_n=(\partial_n\times I_n)\cup(I_n\times\partial_n)$, and $\overline{Q}_n=Q_n\cup\partial Q_n$. We also define the corresponding *dual graphs*: $I_n^*=[0,n+1]\cap\left(\mathbb{Z}+\frac{1}{2}\right)$ and $Q_n^*=I_n^*\times I_n^*$.

For $\beta>0$, we consider the Ising model with inverse temperature β (and with zero magnetic field) on Q_n, with $+$ boundary conditions on $\partial_n\times I_n$ (i.e. the vertical sides) and $-$ boundary conditions on $I_n\times\partial_n$ (i.e. the horizonal sides); let $Z_n^{\pm\pm}(\beta)$ denote the corresponding partition function.

We also consider another Ising model on the same graph, with the same temperature and magnetic field, but with pure $+$ boundary conditions; let $Z_n^{+}(\beta)$ denote the corresponding partition function.

Show that

$$\frac{Z_n^{\pm\pm}(\beta)}{Z_n^{+}(\beta)}$$

is equal to

$$\mathbb{E}\left[\sigma_{\mathrm{sw}}\sigma_{\mathrm{se}}\sigma_{\mathrm{ne}}\sigma_{\mathrm{nw}}\right],$$

where the expectation is taken for an Ising model on Q_n^* with free boundary conditions (and zero magnetic field) at inverse temperature $\beta^*>0$, where β^* is such that $\tanh(\beta^*)=e^{-2\beta}$, and

$$\mathrm{sw}=\left(\frac{1}{2},\frac{1}{2}\right)$$
$$\mathrm{nw}=\left(n+\frac{1}{2},\frac{1}{2}\right)$$
$$\mathrm{ne}=\left(n+\frac{1}{2},n+\frac{1}{2}\right)$$
$$\mathrm{se}=\left(n+\frac{1}{2},\frac{1}{2}\right).$$

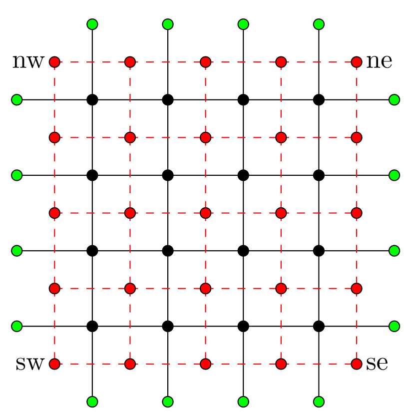


Figure 2. The vertices of Q_n (in black), ∂Q (in green) and Q_n^* (in red) for $n=4$. The edges of $\overline{Q}_n$ are also shown (in black), as are the edges of Q_n^* (in red and dashed).

Exercise 9.

The goal of this exercise is to illustrate a link between simple random walks (SRW) and critical percolation on the faces of the honeycomb lattice.

Let $\triangleright$ be the equilateral triangle with vertices $1, \frac{\sqrt{3}}{2}i, -\frac{\sqrt{3}}{2}i$. For $\delta > 0$, we consider two discretizations of $\triangleright$:

— let $\triangleright_\delta^h$ denote a discretization of $\triangleright$ by a δ-meshed honeycomb lattice.
— let $\triangleright_\delta^s$ denote the discretization of $\triangleright$ by the δ-meshed square lattice $\delta\mathbb{Z}^2$.

We then use $\mathbb{P}_\delta$ to denote the measure associated with critical percolation on the faces of $\triangleright_\delta^h$, and, for $x \in \triangleright$, consider a SRW $(X_n^x)_{n\geq 0}$ on $\delta\mathbb{Z}^2$ starting from a point $x_\delta \in \triangleright_\delta^s$ at minimal distance from x, stopped at the first exit time τ_δ^x of $\triangleright_\delta^s$ (i.e. $\tau_\delta^x = \inf\left\{n : X_n^x \in \delta\mathbb{Z}^2 \setminus \triangleright_\delta^s\right\}$).

Show that for any $z \in \triangleright$, we have

$$\lim_{\delta\to 0} \mathbb{P}_\delta \begin{Bmatrix} z \text{ and } 1 \text{ are separated in } \triangleright_\delta^h \text{ by a} \\ \text{path of black hexagons from } \pm\frac{\sqrt{3}}{2}i \end{Bmatrix} = \lim_{\delta\to 0} \mathbb{E}_\delta \left[\mathfrak{Re}\left(X_{\tau_\delta^z}^z\right)\right].$$

Exercise 10.
Consider the graphs $I_n, I_n^*, Q_n, \partial Q_n, \bar{Q}_n$, and Q_n^* as in Exercise 8.

We first sample a union $\mathcal{S}_T$ of random trees on $\bar{Q}_n$ with the following algorithm:

— Start with $\mathcal{S}_0 = \partial Q_n$.
— As long as there are vertices in $\bar{Q}_n \setminus \mathcal{S}_j$, pick a random vertex in it, and run a loop-erased random walk Λ_j from it to $\mathcal{S}_j$.
— Define $\mathcal{S}_{j+1} = \mathcal{S}_j \cup \Lambda$.
— Stop as soon as $\mathcal{S}_T$ contains all the vertices of $\bar{Q}_n$.

We then take the set of edges $\mathcal{E}$ of Q_n^* that do not cross an edge of $\mathcal{S}_T$. Show that the edges of $\mathcal{E}$ form a uniform spanning tree of Q_n^*.

Bonus question.
What was your favorite topic covered in class?