LATTICE MODELS (MA1, EPFL)

1. Introduction

1.1. **What this Class is About.** The goal of this class is to give an overview of results about lattice models, which should help you understand their structure, what are the kind of questions to ask, and what are the kind of answers to find. This class is a math class: we prove theorems. However it is a bit different to most undergraduate math classes in the sense that we don't emphasize definitions, and that we don't try to construct a theory step by step. Rather: we look at fairly concrete questions, study them, and solve them and only later try to infer what is the general philosophy. The general picture is too deep to be studied axiomatically at this point. One of the goals is to teach what are the key mathematical mechanisms to understand lattice models: how should one analyze them, what should one look for, what techniques work.

1.2. **Lattice Models.** Lattice Models are at the heart of many fields of science:

- Statistical Mechanics: how macroscopic interactions
- Quantum Field Theory: high-energy physics (e.g. Higgs boson)
- Biology: models of cells, etc.
- Ecology: models of growing plants
- Economics: models of interacting agents
- Image procesing: probabilistic models of images
- Machine learning: models of computation

1.3. **Five Topics.** There are mainly five topics that we will discuss in this class, each associated with one or two non-trivial results:

- Simple Random Walk: Recurrence and Transience and PDEs
- Loop-Erased Random Walk and Uniform Spanning Tree: Wilson's Theorem and the Matrix-Tree Theorem
- Percolation: Phase Transition and Cardy's Formula
- Ising Model: Graphical Representations, Sampling and Phase Transition
- Dimer Model: counting dominos

2. Simple Random Walk: Recurrence and Transience

2.1. **What is a Simple Random Walk?** We look here at discrete-time simple random walks $(X_n)_{n\geq 0}$ on a locally finite graph G, that jump at integer times.

- Simple random walk on $\mathbb{Z}$: we start at 0 and at each time, we jump either left or right, with probabilities $\frac{1}{2}$, independently of the past.
- Simple random walk on $\mathbb{Z}^2$: we start at 0, and at each time, we jump on one of the four neighbors, with probabilities $\frac{1}{4}$, independently of the past.
- On a locally finite graph G: start at an origin vertex, and if $X_n = v$, then X_{n+1} is one of the neigbhors of v, chosen with equal probability, independently of the past.

2.2. **What are Basic Interesting Questions?**

- How does the SRW look on the long term?
- How does a SRW trajectory look from far away?
- Does a SRW ever come back to the origin?
- Are there connections with other random objects?

2.3. **A Recent Result.** At this point, it is worth mentioning a beautiful result of Lawler, Schramm and Werner (we won't prove it, but it is nice to know):

- Consider two independent SRW $(X_n)_n$ and $\left(\tilde{X}_n\right)_n$ on $\mathbb{Z}^2$ from $(0,0)$
- Call S_n and $\tilde{S}_n$ the set of points visited by X_n and $\tilde{X}_n$ during the times $k = 1, \ldots, n$ (not including $k = 0$).
- We have that $P_n := \mathbb{P}\left\{S_n \cap \tilde{S}_n = \emptyset\right\}$ decays like $n^{-5/8}$ as $n \to \infty$ (i.e. $\log P_n / \log\left(n^{-5/8}\right) \to 1$).

2.4. **Recurrence and Transience: a Theorem.**

- If we consider the simple random walk on $\mathbb{Z}$ or $\mathbb{Z}^2$, with probability 1
- On $\mathbb{Z}^d$ for $d \geq 3$, with probability > 0 it will not come back to the origin (Theorem attributed to Polya).
- This result is fairly robust (it works on various graphs); in this class we will only look at the square lattice case
- The strategy of the proof: reduce the question to one about the expected number of visits of the origin, which is a Markov chain quantity and we will compute this quantity number using Fourier analysis.

2.5. **The Number of Visits to the Origin.**

- Let N_d be the number of times that a SRW on $\mathbb{Z}^d$ come back to the origin, and π_d the probability of ever coming back, i.e. $\pi_d = \mathbb{P}\{N_d \geq 1\}$.
- Then if $\pi_d = 1$, with probability 1, $N_d = \infty$ and if $\pi_d < 1$ and $\mathbb{E}[N_d] < \infty$ (and of course $N_d < \infty$ with probability 1).
- Why? If $\pi_d = 1$, it is obvious: use the Markov property after the first time back to the origin, i.e. the SRW after that time is just like a 'new' SRW, which will come back.
- We have that $\mathbb{P}\{N_d \geq k\} = (\pi_d)^k$ by the Markov property again, and hence $\sum_{k=1}^{\infty} \mathbb{P}\{N_d \geq k\} = \frac{\pi_d}{1-\pi_d}$ if $\pi_d < 1$.
- Now conclude that $\mathbb{E}[N_d] = \frac{\pi_d}{1-\pi_d} < \infty$ if $\pi_d < 1$ by using an elementary lemma.
- Lemma: if $X \geq 0$ is a random integer the $\mathbb{E}[X] = \sum_{j=1}^{\infty} \mathbb{P}\{X \geq j\}$ (works also if both sides are $+\infty$).
- Proof of the lemma:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k\mathbb{P}\{X = k\} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathbf{1}_{\{j \leq k\}} \mathbb{P}\{X = k\} = \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \mathbb{P}\{X = k\} = \sum_{j=1}^{\infty} P\{X \geq j\}.$$

- So: we need to show that $\mathbb{E}[N_d] = \infty$ if $d \leq 2$ and $\mathbb{E}[N_d] < \infty$ if $d \geq 3$.
- Since $\mathbb{E}[N_d]$ is decreasing with d (why?), we just need to prove the result in dimensions 2 and 3.

2.6. **Markov Chain.**

- For $x, y \in \mathbb{Z}^d$ and $k \geq 0$, let $Q_k(x, y)$ be probability that a SRW starting from x arrives at y after exactly k steps.
- We have the 'matrix-multiplication-like' formula $Q_{k+1}(x, y) = \sum_{z \in \mathbb{Z}^d} Q_k(x, z) Q_1(z, y)$
- Indeed: we need to jump from x to some point z in k step and then from z to y in one step, and both probabilities are independent by the Markov property.
- Let us write $P_k(x)$ for $Q_k(0, x)$. By translation invariance we have $Q_k(x, y) = Q_k(0, y - x)$ and hence $P_{k+1}(x) = \sum P_k(z) P_1(x - z)$.
- Let us write P for P_1. By definition $P(x) = \frac{1}{2d}$ if x is one of the $2d$ neighbors of the origin (i.e. $(\pm 1, 0, \ldots, 0)$, $(0, \pm 1, \ldots, 0)$, ... , $(0, \ldots 0, \pm 1)$) and $P(x) = 0$ otherwise.
- If for two functions $f, g : \mathbb{Z}^d \to \mathbb{C}$, we denote by $f \star g$ the ('convolution') function defined by $(f \star g)(x) = \sum_z f(z) g(x - z)$, then $P_k = P^{\star k}$, where $P^{\star k} := P \star \cdots \star P$ (k times).
- Since we have $\mathbb{E}[N_d] = \sum_{k=1}^{\infty} P_k(0)$, what we need to compute is $\sum_{k=0}^{\infty} P^{\star k}(0)$.
- We would prefer a sum of products (geometric series) to a sum of convolutions. How to transform convolutions into products? Fourier analysis.

2.7. **Fourier Analysis in $d = 1$.**

- A function $f : \mathbb{Z} \to \mathbb{C}$ is nothing but a (bi-infinite) series. Let us assume that everything converges.
- If we a bi-infinite series, we can form a corresponding Fourier series $\mathcal{F}f(x) := \sum_k f(k) e^{ikx}$, which is 2π-periodic function
- We have the classical inversion formula: from $\mathcal{F}f(x)$, we can recover $f(k)$ by $f(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathcal{F}f(x) e^{-ikx} \mathrm{d}x$ (why?).

- If $h = f \star g$, for $f, g : \mathbb{Z} \to \mathbb{C}$, we have that $\mathcal{F}(h) = \mathcal{F}(f)\mathcal{F}(g)$.
- Proof: take $\sum_k f(k) e^{ikx} \sum_\ell g(\ell) e^{i\ell x}$, make a change of variable $\sum_k f(k) e^{ikx} \sum_\ell g(\ell - k) e^{i(\ell-k)x}$ and re-arrange into $\sum_k \left(\sum_\ell f(k) g(\ell-k)\right) e^{ikx}$, which gives $\mathcal{F}h(x)$.
- So for our problem, we have (assuming things converge), we have $\mathcal{F}\left(P^{\star k}\right)(x) = \left(\mathcal{F}(P)\right)^k(x)$ and hence $\mathcal{F}\left(\sum_{k=0}^\infty P^{\star k}\right)(x) = \frac{1}{1-\mathcal{F}(P)}(x)$.
- Since $\mathcal{F}P(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \cos(x)$, we have $\mathcal{F}\left(\sum_{k=0}^\infty P^{\star k}\right)(x) = \frac{1}{1-\cos(x)}$.
- So, if we want to compute $\mathbb{E}[N_{d=1}] = \sum_{k=0}^\infty P^{\star k}(0)$, we use the inversion formula and obtain $\int_{-\pi}^{\pi} \frac{1}{1-\cos(x)} \mathrm{d}x$.
- Since $\cos(x) = 1 - \frac{1}{2}x^2 + \cdots$ near $x = 0$, this integral is divergent, so the expectation is infinite.
- How to make this more rigorous: dampen the sum by adding a mass term $\sum_{k=0}^\infty \lambda^k P^{\star k}(0)$ for $\lambda < 1$, get $\int_{-\pi}^{\pi} \frac{1}{1-\lambda\cos(x)} \mathrm{d}x$, which is finite (and absolutely convergent) and then let $\lambda \to 1$ and use monotone convergence.

2.8. **Fourier Analysis in $d \geq 2$.**

- The analysis is pretty much the same.
- A function $f : \mathbb{Z}^d \to \mathbb{C}$ gives rise to a Fourier series $\mathcal{F}f(\mathbf{x})$ of d variables defined by $\sum_{\mathbf{k}\in\mathbb{Z}^d} f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$ (we wrote $\mathbf{k}\cdot\mathbf{x} := k_1x_1 + \cdots + k_dx_d$), and this function is periodic in each variable.
- The inversion formula is given by $f(\mathbf{k}) = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \mathcal{F}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}x_1 \cdots \mathrm{d}x_d$ (same proof as before).
- For exactly the same reasons as before $\mathcal{F}(f \star g) = \mathcal{F}(f)\mathcal{F}(g)$.
- Hence for our problem, we have as before $\mathcal{F}\left(\sum_{k=0}^\infty P^{\star k}\right)(\mathbf{x}) = \frac{1}{1-\mathcal{F}(P)}(\mathbf{x})$.
- Now $\mathcal{F}(P)(\mathbf{x}) = \frac{1}{d}\sum_{j=1}^d \cos(x_j)$ and hence we get (formally) $\mathbb{E}[N_d] = \left(\frac{1}{2\pi}\right)^d \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{1}{1-\frac{1}{d}\sum\cos(x_j)} \mathrm{d}x_1 \cdots \mathrm{d}x_d$
- Is this integral divergence or convergent? Obviously, the only singularity in the hypercube $[-\pi,\pi]^d$ is at $x_1 = \cdots = x_d = 0$, so let's see what happens there.
- Near $(0,\ldots,0)$, we have $\frac{1}{d}\sum\cos(x_j) = 1 - \frac{1}{2d}\sum_{j=1}^d x_j^2 + \cdots$ and hence we should study t
- In $d = 2$, we can do a polar change of variable, to get $\int_0^\epsilon \int_0^{2\pi} \frac{1}{r^2} r \mathrm{d}r \mathrm{d}\theta$, which is infinite because $\int_0^\epsilon \frac{1}{r}\mathrm{d}r = \infty$, but 'barely infinite'
- In $d = 3$, we can do a spherical change of variable to get $\int_0^\epsilon \int_0^{2\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{r^2} r^2 \cos\varphi \mathrm{d}\varphi \mathrm{d}\theta \mathrm{d}r$, which is obviously finite.
- To make things rigorous, exact same trick as in 1D.

3. Simple Random Walks and PDEs

3.1. **Goal of this lecture.**

- Explain the connection between: simple random walks and the Laplace equation $\Delta f = 0$, and the heat equation $\partial_t f = \Delta f$.
- This connection is useful to
  - study the continuous limit of random walks (in particular, calculate things on it)
  - understand Laplace equation and the heat equation (in particular, obtain qualitative results easily)

3.2. **One-Dimensional Problem.**

- We have a segment $\{1,\ldots,n-1\} \subset \mathbb{Z}$, with boundary $\{0,n\}$ and $B = \{n\}$.
- Consider $H(k)$, the chance that a SRW from $k \in \{0,\ldots,n\}$ hits n before 0? Obviously $H(0) = 0$ and $H(n) = 1$.
- Now we have a relation for inner vertices: for $k \in \{1,\ldots,n-1\}$,we have $H(k) = \frac{1}{2}(H(k-1) + H(k+1))$, which is called the mean value property.
- Why? The SRW jumps to either $k+1$ or $k-1$ with probabilities $\frac{1}{2}, \frac{1}{2}$. Once it is there, its chances to hit 0 are $H(k-1)$ and $H(k+1)$ by Markov property.
- How many equations do we have ? $n-1$ linear equations. How many unknowns? The values of H at $\{1,\ldots,n-1\}$, so $n-1$, too. So, it looks like we have enough.
- Can we guess a solution? A linear function with the appropriate boundary conditions is natural: $H(k) = \frac{k}{n}$.
- Can we argue there are no other solutions? If we had two, we could take their difference D, which would satisfy $D(0) = D(n) = 0$ and the mean value property.
- Why must $D = 0$? A simple argument, called *maximum principle*: if it weren't equal to 0, then it would have either a strict maximum or a strict minimum on $\{1,\ldots,n-1\}$.

- But that would contradict the mean value property (how can a strict maximum/minimum be equal to the values of the two neighbors?).

3.3. Discrete Harmonic Measure in General.

- Take a graph $G \subset \mathbb{Z}^d$ with boundary ∂G, fix a subset of the boundary $B \subset \partial G$. To visualize: think for instance $d = 2$, G is a rectangular box, the subset B is the bottom of the box.
- Take a SRW starting from a vertex x of G, and stop it when it hits ∂G. What is the chance $H(x, B)$ that it has hit B?
- Obviously $H(x, B) = 1$ if $x \in B$ and $H(x, B) = 0$ if $x \in \partial G \setminus B$. What about the interior of G?
- For the same reasons as in one-dimension, we have that $H(x, B)$ is *discrete harmonic*, i.e. it satisfies the mean-value property $H(x) = \frac{1}{2d} \sum_{y \sim x} H(y)$ for all $x \in G$.
- And for the same reasons as before (maximum principle), $H(\cdot, B)$ is the unique discrete harmonic function that is equal to 1 on B, 0 on $\partial G \setminus B$.

3.4. Discrete Dirichlet Problem.

- Given a graph G with boundary ∂G and a function $f : \partial G \to \mathbb{R}$, what is the unique (why?) discrete harmonic function $F : G \cup \partial G \to \mathbb{R}$ such that $F|_{\partial G} \equiv f$?
- This kind of problem is particularly useful to extend a function on ∂G in a natural manner:
  - for instance if $f : \partial G \to \mathbb{R}$ is the value of an eletric potential on a grid G, then F is the value of the potential inside
  - another (related) example: if we put some temperature on the boundary of a grid and wait until the temperature equilibrates inside, it will be given by F
- Answer: for a SRW $(X_k^x)_{k \geq 0}$ from $x \in G \cup \partial G$, let τ^x be the first $k \geq 0$ such that $X_k^x \in \partial G$, i.e. the first time the random walk exits G. Then $F(x) = \mathbb{E}[f(X_{\tau^x}^x)]$.
- Why? F has the correct boundary values (obvious) and is discrete harmonic because of the Markov property.
- Hence: the space of discrete harmonic functions on $G \cup \partial G$ and the space of functions on ∂G have the same dimension and are in one-to-one correspondence (why?).

3.5. Discrete Dirichlet Laplacian.

- Let us consider a graph G with boundary ∂G and a fixed function $f : \partial G \to \mathbb{R}$, called Dirichlet boundary condition.
- The Laplacian with boundary condition f is the linear map $\Delta_G^f : \mathbb{R}^G \to \mathbb{R}^G$ defined by $\Delta_G^f u(x) := \frac{1}{2d} \sum_{y \sim x} (u(y) - u(x))$, where $u(y) := f(y)$ for each $y \in \partial G$.
- For any fixed $G, \partial G, f$ the Laplacian Δ_G^f is injective by the maximum principle and hence it is invertible by dimensionality.
- Actually (exercise), the discrete Laplacian with 0 boundary conditions is negative definite and so is its inverse.
- The quadratic form $E(u) := -u \cdot \Delta_G^0 u$ (where $\cdot$ is the usual dot product) is called the Dirichlet energy and it is minimized by the solution to the Dirichlet problem.

3.6. Discrete Dirichlet Green's Function and Poisson's equation.

- Consider a finite graph G and its discrete Laplacian $\Delta := \Delta_G^0$. We know Δ is invertible. What is its inverse?
- Consider for each $x \in G$, the SRW $(X_k^x)_{k \geq 0}$ from x, the stopping time τ^x (the first time that X_k exits G) and define the function $U(x, y) := \mathbb{E}\left[\sum_{k=0}^{\tau^x - 1} \mathbf{1}_{X_k = y}\right]$, i.e. the number of visits of y starting from x before hitting ∂G.
- We have that $(\Delta U(\cdot, y))(x) = 0$ for all $y \neq x$ by the Markov property (the SRW doesn't visit y at time 0) and $(\Delta U(\cdot, y))(y) = -1$ (the time 0 visit of y lost after jumping away by one step).
- What does this mean? If we write the operators in a matrix notation, it means that $(\Delta U)_{x,y} = -\delta_{x,y}$, i.e. $\Delta U = -\mathrm{Id}$, i.e. $\Delta^{-1} = -U$.
- This allows one to solve the discrete Poisson equation $\Delta u = g$, $u|_{\partial G} \equiv 0$ for $g : G \to \mathbb{R}$ given: we have that $u = -Ug$ is the solution.

3.7. Discrete Heat Equation.

- Consider a finite graph G, with boundary ∂G. Consider a SRW $(X_k^x)_{k\geq 0}$ as usual. Let $\Delta := \Delta_G^0$
- Set $H(x,t) := \mathbb{P}\{X_k^x \in G \, \forall k \leq t\}$, i.e. the probability that a SRW from x stays in G until time t at least.
- We have that $\sum_{y\sim x} \frac{1}{4} H(y,t) = H(x,t+1)$ and hence $\Delta H = [d_t] H$, where $[d_t] F := F(t+t) - F(t)$ for $F : \mathbb{N} \to \mathbb{R}$ and we take Δ and $[d_t]$ with respect to the relevant variable (space and time respectively).
- The boundary conditions are $H(\cdot|_{\partial G}, t) = 0$ for all $t \geq 0$ and $H(\cdot|_G, 0) = 1$.
- The solution is obviously unique in this case.

4. Convergence

4.1. Continuous Operators.

- Let us now consider the grid $\delta\mathbb{Z}^d$, i.e. the lattice rescaled by a factor $\delta > 0$, called the mesh size: the distance between two adjacent vertices becomes δ. What happens as $\delta \to 0$?
- Let us look at a smooth bounded domain $G \subset \mathbb{R}^d$ and at its discretization $G_\delta := G \cap \delta\mathbb{Z}^d$. Whenever needed, identify points of G with the closest vertices of G_δ.
- We have that if $F : G \to \mathbb{R}$ is a $\mathcal{C}^2$ function then $\frac{1}{\delta^2}\Delta_{G_\delta} F(x) \to \Delta F(x)$ for all $x \in G$, where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. Why? Taylor expand F.

4.2. Convergence of Solutions.

- There are many questions and many answers of this type. Let us ask the most natural one: consider a family of discrete approximations $(G_\delta)_{\delta>0}$ of a smooth domain of $\mathbb{R}^d$ and fix piece of the boundary $B_\delta := \partial G_\delta \cap \Lambda$ for an open set $\Lambda \subset \mathbb{R}^d$.
- Statement: as $\delta \to 0$ the harmonic measure $H_{G_\delta}(z, B_\delta)$ converges to $H_G(z, B)$, where $H_G(\cdot, B)$ is continuous harmonic.
- There are many other similar statements, which can be proven with similar techniques.

4.3. Convergence Strategy.

- Interpolation: we want to use Arzelà-Ascoli precompactness results, so we first interpolate our functions
- Regularity estimates: we need to make sure that the function is not 'too crazy', i.e. 'it doesn't wiggle too much'
- A priori boundary estimates: make sure that as we approach the parts of the boundary where the harmonic measure is 0, we get close to 0 and similarly as we get close to 1
- Uniqueness result: there is a unique continuous harmonic function F with the relevant boundary conditions.
- Classical conclusion: we can apply Arzelà-Ascoli to make sure that we have convergence along subsequences, make sure it has the relevant boundary values and reason by contradiction to ensure convergence if we have uniqueness.

4.4. A priori estimates.

- The idea is to use a discrete Harnack-type inequality to control the discrete partial derivatives of discrete harmonic functions.
- For a function $f : \delta\mathbb{Z}^d \to \mathbb{R}$, and $k \in \{1, \ldots, d\}$, we denote by $\partial_\delta^k f$ the partial derivative defined by $\partial_\delta^k f(x) = f(x + \delta e_k) - f(x)$, where e_k is the k-th vector of the canonical basis.
- The Harnack inequality: for a discrete harmonic function f defined on $B_\delta(x) := \left\{z \in \mathbb{Z}^d : |z - x| \leq r\right\}$, there exists $C > 0$ such that for all $k \in \{1 \ldots, d\}$

$$\left|\partial_\delta^k f(x)\right| \leq C \max_{z \in B_\delta(x,r)} |f(z)|$$

4.5. Boundary control.

- We already know our functions are between 0 and 1 and behave nicely inside; what we need to know is that they don't suddenly jump as we approach the boundary
- First, notice that they *are* going to jump when we go from B_δ to its complementary $C_\delta := \partial G_\delta$
- So, let us just look at $B_\delta^r := \{z \in B_\delta : \mathrm{d}(z, C_\delta) \geq r\}$ and $C_\delta^r := \{z \in C_\delta : \mathrm{d}(z, B_\delta) \geq r\}$ for some small fixed $r > 0$.
- What we want to show (r is fixed): for any $\epsilon > 0$, there exists $\rho > 0$ such that for all $\delta > 0$ and $z \in G_\delta$ with $\mathrm{d}(z, B_\delta^r) \leq \rho$ then $|H_{G_\delta}(z, B_\delta) - 1| \leq \epsilon$ and for all $z \in G_\delta$ with $\mathrm{d}(z, C_\delta) \leq \rho$ then $|H_\delta(z, B_\delta)| \leq \epsilon$.

- Let us prove this in 2D: the inequality is called the discrete Beurling estimate.

4.6. **Discrete Beurling estimate in 2D.**

- The discrete Beurling estimate is an explicit form of the control that we need to deal the harmonic measure H_δ near boundary value points: it states that there exists constants $C, \alpha > 0$ such that for all z's

$$|H_\delta(z, B_\delta)| \leq C \left(\frac{\mathrm{d}(z, B_\delta)}{\mathrm{d}(z, C_\delta)} \right)^\alpha$$

- The strong discrete Beurling estimate (which we will not prove in this class) gives that the optimal (i.e. biggest) universal α that can be chosen is $\alpha = \frac{1}{2}$.
- The prove it, we need to prove that with (polynomially) high probability a SRW starting from z will hit B_δ before C_δ: we must find a collection of of independent 'challenges' that the SRW must overcome.
- We can assume $\mathrm{d} := \mathrm{d}(z, B_\delta) < \mathrm{d}(z, C_\delta) := \mathrm{D}$ (otherwise there is nothing to prove) and construct $k = \lfloor \log_2 (D/d) \rfloor$ concentric annuli around z of inner radii $\mathrm{d}, 2\mathrm{d}, 4\mathrm{d} \ldots, 2^k \mathrm{d} \leq D$ and outer radii $2\mathrm{d}, 4\mathrm{d}, 8\mathrm{d}, \ldots$
- It is easy to see (but a bit tedious to write) that there exists a constant $\kappa > 0$ such that for each such annulus, the probability that a SRW entering from the inner boundary closes a loop around the inner radius before hitting the outer boundary is at least κ.
- Since each of these annuli intersects B_δ, the probability at a SRW starting from z hits C_δ before B_δ is at most $(1-\kappa)^k \leq (\mathrm{d}/\mathrm{D})^\alpha$ for some $\alpha > 0$: if the SRW completes a loop in any of those, then it hits B_δ.

4.7. **General Case.**

- In dimension greater than 2, there is no discrete Beurling estimate, so one must add conditions about the boundary in order not to have strange behavior.
- If for instance, the boundary is smooth, one can argue that the neighborhood of each boundary point looks like a half-space and use Beurling-like arguments.

4.8. **Discrete Harnack Inequality.**

- Let us prove the 2D Harnack inequality: the result and the proof are essentially the same in all dimensions.
- Consider the discretization $\mathbb{D}_\delta$ of a disk $\mathbb{D} = \{|z| < r\}$ by $\delta\mathbb{Z}^2$ and identify the points of the plane with complex numbers
- Let us show the following inequality: for any any discrete harmonic function $f : \mathbb{D}_\delta \to \mathbb{R}$, we have

$$|f(i\delta) - f(-i\delta)| \leq \mathrm{Cst} \cdot \delta \cdot \max_{z \in \partial \mathbb{D}_\delta} |f(z)|$$

- Similar arguments allow to similarly bound $|f(\delta) - f(0)|$ and $|f(i\delta) - f(0)|$
- The idea is to represent $f(\pm i\delta)\ \mathbb{E}\left[f\left(X_{\tau^\pm}^{\pm i\delta}\right)\right]$ for SRWs starting from $\pm i\delta$ where $\tau^\pm$ are the hitting times of $\partial\mathbb{D}_\delta$ and compare the expectations.
- The idea to compare the expectations is to *couple* the SRWs $\left(X_n^{i\delta}\right)_{n\geq 0}$ and $\left(X_n^{-i\delta}\right)_{n\geq 0}$ in such a way that most of the time, they hit $\partial\mathbb{D}_\delta$ at the same place.
- The coupling is the following: sample $\left(X_n^{-i\delta}\right)_{n\geq 0}$ by taking the mirror image with respect to $\mathbb{R}$ of $\left(X_n^{i\delta}\right)_{n\geq 0}$ until the first time ς that $X_\varsigma^{i\delta} \in \mathbb{R}$, after which we set $X_n^{-i\delta} = X_n^{i\delta}$.
- It is easy to see that this is a coupling (i.e. that $X_n^{-i\delta}$ is a SRW) and that for this coupling

$$|f(i\delta) - f(-i\delta)| \leq \mathbb{P}\left\{\tau^+ < \varsigma\right\} \max_{z \in \partial\mathbb{D}_\delta} |f(z)|.$$

- We now just need to bound $\mathbb{P}\{\tau^+ < \varsigma\}$, i.e. the chance that $X_n^{i\delta}$ hits $\partial\mathbb{D}_\delta$ before $\mathbb{R}$ by $\mathrm{Cst} \cdot \delta$.
- This can indeed be compared up to constant to the chance that a SRW in a strip $\delta\mathbb{Z}^2 \cap \{0 < \Im\mathfrak{m}(z) < 1\}$ hits the top before the bottom.
- That probability equals δ by the previous section (by translational invariance of the strip, this becomes a 1D problem).

4.9. **Uniqueness.**

- Now take $\delta \to 0$ and then $r \to 0$, we get a function on G, which is smooth, has boundary value 1 on B and boundary value 0 on $\partial G \setminus B$; is it unique?
- As usual, by linearity we just need to argue that we have a continuous maximum principle
- Observe that if $\Delta f(x) > 0$, then f can't have a maximum (even a non-strict one) at x: otherwise the Hessian $Hf|_x$ would be negative semi-definite, and $\Delta f(x) = \mathrm{Tr}\left(Hf\big|_x\right) \leq 0$, a contradiction.

- How can we then argue that f cannot have a maximum? The idea is to reason by contradiction: if there were a maximum, the eigenvalues of the Hessian would have to be non-positive, and their sum (the Laplacian) could not be positive.
- Similarly, if $\Delta f(x) < 0$, then f can't have a minimum.
- How do we get that if $\Delta f = 0$, there is no > 0 maximum.
- Suppose without loss of generality there were a maximum of value 1000 inside, that the domain were of radius 1. If we added the function $\|x\|^2$ to f, we still would have that $f + \|x\|^2$ has a maximum inside (we would be equal to at most 1 on the boundary, and at least 1000 inside). But that function would now have a positive Laplacian.
- The same works to rule out a < 0 minimum.

4.10. **Two-Dimensional Harmonic Functions.**

- We can look at complex-valued functions $f : \mathbb{C} \to \mathbb{C}$ as functions $F : \mathbb{R}^2 \to \mathbb{R}^2$.
- What can we say about holomorphic functions as functions of two real variables? They satisfy Cauchy-Riemann equations $\partial_1 F_1 - \partial_2 F_2 = 0$, $\partial_2 F_1 + \partial_1 F_2 = 0$.
- It is easy to see that holomorphic functions are actually harmonic: $\partial_{11} F_1 = \partial_{12} F_2 = \partial_{21} F_2 = -\partial_{22} F_1$ and $\partial_{11} F_2 = -\partial_{22} F_2$ for the same reason, so $\Delta F = 0$.
- Conversely, it can be shown that any real-valued harmonic function on a simply-connected domain (i.e. with no holes) is the real part of a holomorphic function.
- We can use this to find explicit harmonic functions; some obvious properties (composition of holomorphic functions is holomorphic) are going to be useful

4.11. **Exact Calculations.**

- An example of a nice question that can be solved using the above method: consider the unit disk $\mathbb{D} := \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}$ and discretize it by $\mathbb{D}_\delta$
- What is the chance that a SRW from (x, y) exits $\mathbb{D}_\delta$ through the upper-right quadrant as $\delta \to 0$?
- What we want to find: a function that is a harmonic on $\mathbb{D}$ and equals 1 on $\left\{e^{i\theta} : \theta \in [0, \pi/4]\right\}$ and 0 on the rest of $\partial\mathbb{D}$.
- A conformal mapping to the right half plane $\mathbb{H}_E := \{z \in \mathbb{C} : \mathfrak{Re}(z) > 0\}$ is $\frac{z+1}{z-1}$ and its inverse is $\frac{z-1}{z+1}$
- Another example: it can be shown that the function $z \mapsto \frac{2\sqrt{2\pi}}{\Gamma^2(1/4)} \int_0^z \frac{1}{\sqrt{\zeta(\zeta^2-1)}} \mathrm{d}\zeta$ maps $\mathbb{H}$ to the square $[-1, 0] \times [0, i]$ (with $-1 \mapsto 1$, $0 \mapsto 0$, $1 \mapsto i$, and $\infty \mapsto -1 + i$).
- If we want to find 1/2-level line for the probability that a SRW hits the top of the box, we can look at the image of a circle on $\mathbb{H}$ via that mapping.

5. Uniform Spanning Tree

5.1. **Loop-Erased Random Walk.**

- A loop-erased random walk (LERW) is built by applying a loop-erasure procedure to a simple random walk.
- The so-called chronological loop-erasure procedure is as follows: whenever the closes a loop, i.e. arrives at time n to a vertex where it has come before at time t_n, the list of points visited between times t_n and n is erased.

5.2. **Spanning Trees and Wilson's algorithm.**

- A spanning tree of a connected graph G is a subgraph of G that is a tree (connected, no cycle) and that contains all the vertices of G.
- There is a finite number of spanning trees of a finite graph, and this number is given by the matrix-tree theorem (homework)
- Wilson's algorithm is a very efficient way to sample a spanning tree using by adding branches using LERW.
- The idea is to choose a root x_0 and to construct a growing family of trees $\{x_0\} = T_0 \subset T_1 \subset \cdots \subset T_k$ such that $T_{i+1} \setminus T_i$ is a LERW from an arbitrary vertex of $x_i \in G \setminus T_i$ stopped upon hitting T_i and stopping when we get a spanning tree.
- Obviously, we get a spanning tree with this method. The question is: why is the measure uniform?
- The idea is to construct a probability space made of 'stacks of arrows' that generates the tree and the loops that were erased in the LERW

- This will show that the measure is uniform and the tree is actually independent of the choices of $x_0, \ldots, x_r$.

5.3. **Stack of Cycles.**

- To each vertex $x \in G \setminus \{x_0\}$, associate an infinite stacks of random 'arrows' that points to the neighbors of x, uniformly, independently of each other.
- These arrows can be used to sample a SRW from any vertex $x \in G \setminus \{x_0\}$, stopped upon hitting x_0: when we are at a vertex jump to the neighbor pointed by the arrow at the top of the stack and remove it.
- If we look at the arrows at the top of the stacks, they form a number of cycles, plus a tree pointing to the root x_0
- We can take the arrows of a cycle at the top of the stack and remove it: we call this a 'cycle removal' procedure
- If we keep removing cycles until we don't have any (this will happen with probability one), we get a tree
- What we will show: the order in which we remove the cycles is irrelevant and the tree that we get is sampled with uniform probability

5.4. **Cycle Lemma.**

- The order in which we remove the cycles is essentially irrelevant: if there is a cycle $\mathcal{C}$ that we can remove at some point and we choose to remove some other cycles $\mathcal{C}_1 \ldots, \mathcal{C}_k$, we can still remove $\mathcal{C}$ later
- Indeed, if there is another cycle, either it lives on disjoint vertices and then it doesn't affect the fact that $\mathcal{C}$ can be removed later, or it has some intersection.
- If there is some intersection, let us consider the first $\mathcal{C}_j$ with some intersection: we will show that actually $\mathcal{C} = \mathcal{C}_j$.
- Indeed, take a vertex v in the intersection, and follow the arrow at the top of the stack: it must be the same for $\mathcal{C}$ and $\mathcal{C}_j$ (because they are taking the first arrow from the stack above v).
- But then the next vertex (following the arrow) is also the same for $\mathcal{C}$ and $\mathcal{C}_j$, and so on and so forth. So we get $\mathcal{C} = \mathcal{C}_j$.

5.5. **Probability Measure.**

- Once we understand the picture (a collection of loops and a tree just below), it is fairly easy to show that the tree that we get is uniform
- Let us look at a possible 'history', i.e. a set of loops sitting on top of a tree. It has the same probability as an alternate history with the same set of loops on top of any different tree.
- To compute the probability of a tree, we sum over all the possible 'histories' leading to it: we deduce from the above remark that each tree has the same probabiity.

6. Percolation

6.1. **Critical percolation.**

- We consider critical percolation on the honeycomb lattice: each hexagon is colored in black/white, with probability $\frac{1}{2}/\frac{1}{2}$, independently of the others.
- Percolation is originally a model of a porous medium: we can think that black hexagons are filled with matter, while white ones are empty.
- It is natural to ask about connectivity questions for that medium: is there a path made of white hexagons joining two sets?
- The probability $\frac{1}{2}/\frac{1}{2}$ is critical [Kes80]: if one color is slightly favored over the other, then we will have an infinite connected component of that color.

6.2. **Cardy's formula.**

- Consider a quad (Ω, a, b, c, d) (i.e. domain with $\partial\Omega$ a simple curve and four points $a, b, c, d \in \partial\Omega$ in ccw order), a discretization $(\Omega_\delta)_{\delta>0}$ by a honeycomb domain of mesh size δ, and identify a, b, c, d with the closest boundary vertice.
- Crossing probabilities: what is the chance $\mathbb{P}_{\Omega_\delta}\{[ab] \leftrightsquigarrow_w [cd]\}$ that there is a path of white hexagons linking $[ab]$ to $[cd]$?
- Conformal invariance of crossing probabilities (Smirnov, 2001): if (Ω, a, b, c, d) and $\left(\tilde{\Omega}, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\right)$ are conformally equivalent (i.e. there exists a conformal map $\Omega \to \tilde{\Omega}$ with $a, b, c, d \mapsto \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$), then

$$\lim_{\delta\to 0} \mathbb{P}_{\Omega_\delta}\{[ab] \leftrightsquigarrow_w [cd]\} = \lim_{\delta\to 0} \mathbb{P}_{\tilde{\Omega}_\delta}\left\{\left[\tilde{a}\tilde{b}\right] \leftrightsquigarrow_w \left[\tilde{c}\tilde{d}\right]\right\}.$$

- There exists an explicit formula (Cardy's formula) for the equilateral triangle: if Δ is an equilateral triangle with vertices a, b, c, then $\lim_{\delta\to 0}\mathbb{P}_{\Delta_\delta}([ab] \leftrightsquigarrow [cd]) = |c-d| / |c-a|$.

6.3. Riemann's mapping theorem.

- Let Ω and $\tilde{\Omega}$ be two Jordan domains (i.e. $\partial\Omega$ and $\partial\tilde{\Omega}$ are closed simple curves), and let $a, b, c \in \partial\Omega$ and $\tilde{a}, \tilde{b}, \tilde{c} \in \partial\tilde{\Omega}$ be distinct boundary points, in counterclockwise (ccw) order.
- Then there exists a unique conformal mapping $\varphi : \Omega \to \tilde{\Omega}$ with $a, b, c \mapsto \tilde{a}, \tilde{b}, \tilde{c}$.
- This statement can be generalized to arbitrary simply-connected domains, provided the boundary points are replaced by prime ends.

6.4. Cardy's formula and main statement.

- For a Jordan domain (Ω, a_1, a_2, a_3), there exists a unique conformal mapping φ from Ω to the equilateral triangle $\triangle$ with vertices $1, \pm\frac{\sqrt{3}}{3}i$, with $a, b, c \mapsto 1, \frac{\sqrt{3}}{3}i, -\frac{\sqrt{3}}{3}i$.
- Carleson's formulation of Cardy's formula [Smi01]: $\lim_{\delta\to 0}\mathbb{P}_{\Omega_\delta}\{[a_1a_2] \leftrightsquigarrow_w [a_3a_4]\} = \mathfrak{Re}(\varphi(a_4))$.
- By uniqueness of the conformal mapping $\varphi : \Omega \to \Delta$, this proves the conformal invariance.
- To prove Cardy's formula, we prove the following statement:
  - Set $A_1 := [a_2a_3]$, $A_2 := [a_3a_1]$ and $A_3 := [a_1a_2]$. For a vertex $z \in \Omega_\delta$ define $H_\delta^1(z)$, $H_\delta^2(z)$ and $H_\delta^3(z)$ by
  $$H_\delta^\mu(z) := \mathbb{P}_{\Omega_\delta}\{a_\mu \text{ and } z \text{ are separated from } A_\mu \text{ by a white path}\}.$$
  - We set $\varphi_\delta := H_\delta^1 + \frac{\sqrt{3}}{2}iH_\delta^2 - \frac{\sqrt{3}}{2}iH_\delta^3$. The main statement reads
  $$\varphi_\delta \underset{\delta\to 0}{\longrightarrow} \varphi.$$
  - Evaluating with $a_4 \in [a_3a_1]$, we get Cardy's formula.

6.5. Strategy for the proof.

- Precompactness: we want to show that $(\varphi_\delta)_{\delta>0}$ is uniformly equicontinuous on $\overline{\Omega}$. Let φ be a subsequential scaling limit $\varphi_{\delta_n} \to \varphi$ (uniqueness of the limit will yield the full convergence).
- Boundary conditions: we show that if $\varphi_{\delta_n} \to \varphi$, then φ is a homeomorphism $\partial\Omega \to \partial\triangle$ with $a_1, a_2, a_3 \mapsto 1, \frac{\sqrt{3}}{3}i, -\frac{\sqrt{3}}{3}i$.
- Analyticity: we show that if $\varphi_{\delta_n} \to \varphi$, then φ is analytic. This will follow from approximate discrete Cauchy-Riemann relations.
- To conclude, i.e. show that $\varphi : \Omega \to \triangle$ is a bijection inside Ω. We use the argument principle, take $w \in \mathbb{C}$ and want to show that $\varphi(z) - w$ has a single zero in Ω iff $w \in \triangle$:
$$\#\{\text{zeros of } \varphi(z) - w\} = \frac{1}{2\pi i}\oint_{\partial\Omega}\frac{\varphi'(z)}{\varphi(z) - w}\mathrm{d}z = \frac{1}{2\pi i}\oint_{\partial\triangle}\frac{1}{\zeta - w}\mathrm{d}\zeta = \mathbf{1}_\triangle(w).$$

6.6. Symmetry and self-duality.

Like for the random walks, we first need some simple a priori estimates for the precompactness part.

- Suppose that the honeycomb lattice has edges parallel to $1, e^{\pm 2\pi i/3}$ and consider the square $S = [0,1] \times [0,i]$, and discretize S by a symmetric honeycomb domain S_δ, with $\delta > 0$ small.
- We have $\mathbb{P}_{S_\delta}\{[0,i] \leftrightsquigarrow_w [1, 1+i]\} = \frac{1}{2}$: either we have a horizontal white path or a vertical black path crossing, both events have the same probability. This works on any symmetric domain.

6.7. RSW estimate.

- Consider the rectangle $R = [0,2] \times [0,i]$ and a symmetric discretization R_δ. We would like to show the Russo-Seymour-Welsh (RSW) estimate: we have$\mathbb{P}_{R_\delta}\{[0,i] \leftrightsquigarrow_w [2, 2+i]\} \geq \frac{1}{16}$.
- To prove RSW, we will want to paste white paths together. We need the Fortuin-Kasteleyn-Ginibre (FKG) inequality:
  - For $\mathcal{A}$ and $\mathcal{B}$ events of the type 'there is a white path from here to there', then $\mathbb{P}(\mathcal{A}\cap\mathcal{B}) \geq \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B})$, or equivalently $\mathbb{P}(\mathcal{A}|\mathcal{B}) \geq \mathbb{P}(\mathcal{A})$.
  - The latter is intuitive, because the existence of a white path somewhere can only increase the chances of seeing a white path elsewhere.
- With FKG, we can prove RSW:
  - With probability $\frac{1}{2}$, there is a path $\gamma : [0,i] \leftrightsquigarrow [1, 1+i]$, made only of white hexagons, take the lowest such path possible. Whatever is above it is independent percolation.

  - Let $\tilde{\gamma}$ be the reflection of γ with respect to the line $1+i\mathbb{R}$ and let D_δ be the connected component $R_\delta \setminus (\gamma \cup \tilde{\gamma})$ lying above $\gamma \cup \tilde{\gamma}$ and intersecting the line $1+i\mathbb{R}$.
  - With probability $\frac{1}{2}$, there is a path $\lambda \subset D_\delta$ linking the bottom-left part of ∂D_δ to the top-right one, again by symmetry. Joining γ and λ yields a white path $[0,i] \leftrightsquigarrow [1+i, 2+i]$.
  - So, we have $\mathbb{P}_{R_\delta}(\mathcal{A}) \geq \frac{1}{4}$, where $\mathcal{A} := \{[0,i] \leftrightsquigarrow_w [1+i, 2+i]\}$. By symmetry, we have $\mathbb{P}_{R_\delta}(\mathcal{B}) \geq \frac{1}{4}$, where $\mathcal{B} := \{[2, 2+i] \leftrightsquigarrow_w [i, 1+i]\}$.
  - If both $\mathcal{A}$ and $\mathcal{B}$ occur, there is a white path $[0,i] \leftrightsquigarrow [2, 2+i]$. By FKG, $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \geq \frac{1}{16}$, which is the desired result.
- With FKG and RSW, the probability of a crossing in the discretizations of rectangles $[0, L] \times [0, i]$ are uniformly bounded from below with respect to $\delta > 0$ (and from above by duality):
  - We can paste several crossings which exist with positive probability from RSW.

6.8. Annulus crossing estimate.

- By FKG and RSW, we get that the probability of a white loop in an annulus of inner and outer radii 1 and 2 is uniformly bounded from below with respect to δ (we paste again).
- For $r, R > 0$ consider the discretization A_δ of an annulus of inner radius r and outer radius R. The probability that a black path links the inner circle to the outer circle is bounded by $C\left(\frac{r}{R}\right)^\alpha$, for universal $\alpha, C > 0$.
  - To prove that, we decompose A_δ into $k := \lfloor \log_2\left(\frac{R}{r}\right) \rfloor$ concentric annuli of inner and outer radii $2^{j-1}r$ and $2^j r$ for $j = 1, \ldots, k$
  - For each annulus, there is a uniformly positive chance of a white crossing in it.

6.9. Precompactness.

- To show precompactness, we will show that $\left(H^1_\delta\right)_\delta$ is uniformly Hölder continuous (the same reasoning applies to H^2_δ and H^3_δ)
  - There exists $C > 0$ and $\alpha > 0$ such that for any $x, y \in \Omega_\delta$, $\left|H^1_\delta(x) - H^1_\delta(y)\right| \leq C \mathrm{d}_\Omega(x,y)^\alpha$, where $\mathrm{d}_\Omega(x,y)$ is the length of the shortest path from x to y in $\overline{\Omega}$
- How to prove this? Let us assume x and y close (otherwise, there is nothing to prove).
  - We have that (writing $\|_w$ for 'there is a white path separating')

$$\begin{aligned} H^1_\delta(x) - H^1_\delta(y) &= \mathbb{P}_{\Omega_\delta}\{a_1, x \|_w A_1\} - \mathbb{P}_{\Omega_\delta}\{a_1, y \|_w A_1\} \\ &= \mathbb{P}_{\Omega_\delta}(\{a_1, x\|_w A_1\} \setminus \{a_1, y\|_w A_1\}) - \mathbb{P}(\{a_1, y\|_w A_1\} \setminus \{a_1, x\|_w A_1\}). \end{aligned}$$

  - Let us study the probability of the event $E_1(x,y) = \{a_1, x\|_w A_1\} \setminus \{a_1, y\|_w A_1\}$.
    - We see that the occurence of E_1 implies the existence of a white path from A_2 to A_3 passing between x and y, and a black path from A_1 to A_1 separating x and y.
    - In turn, each such path, implies that a white or a black path goes from a 'microscopic' circle (i.e. of radius $\mathrm{d}_\Omega(x,y)$) to a macroscopic circle (i.e. of radius $\mathrm{dist}(\{x,y\}, A_j)$ for $j = 1, 2, 3$).
    - By topology argument, at least one of the macroscopic circles is 'big' (i.e. greater than a uniform ϵ), and we can bound the probability this path by the application of RSW above.
  - Hence, we get $\mathbb{P}(E_1(x,y)) \leq C \mathrm{d}(x,y)^\alpha$, and by symmetry, we deduce the Hölder-continuity.
- To extract converging subsequences, we extend φ_δ into a continuous function (by piecewise affine interpolation for instance), and use Arzelà-Ascoli: obviously $(\varphi_\delta)_\delta$ is bounded and equicontinuous.
- From now on, we will assume that φ is a subsequential scaling limit $\lim_{\delta_n \to 0} \varphi_{\delta_n}$, with $\varphi = H^1 + i\frac{\sqrt{3}}{3}H^2 - i\frac{\sqrt{3}}{3}H^3$.

6.10. Boundary conditions.

- We want to prove that φ is a homeomorphism $\partial\Omega \to \partial\Delta$, so we want to prove that it is a homeomorphism $A_1 \to \left[\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3}\right]$, $A_2 \to \left[-\frac{i\sqrt{3}}{3}, 1\right]$, $A_3 \to \left[1, \frac{i\sqrt{3}}{3}\right]$.
- To prove that $\varphi : A_1 \to \left[\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3}\right]$ is a homeomorphism, we should prove that
  - For any $z \in A_1$ $H^1(z) = 0$. This follows from RSW:
    - If $z \in \Omega_\delta$ is at microscopic distance from A_1, then if $Q^1_\delta(z)$ happens, there is a white path from z to A_2 and to A_3
    - At least one of A_2 and A_3 is at macroscopic distance from z. Hence $\mathbb{P}_{\Omega_\delta}\left(Q^1_\delta(z)\right) \to 0$.
  - For any $z \in A_1$ $H^2(z) + H^3(z) = 1$. This follows essentially from self-duality:

    * For $z \in \Omega_\delta$, $\mathbb{P}_{\Omega_\delta}\left(Q_\delta^2(z)\right) = \mathbb{P}_{\Omega_\delta}\left(\tilde{Q}_\delta^2(z)\right)$, where $\tilde{Q}_\delta^2 = \{a_2, z \parallel_{\mathrm{b}} A_2\}$.
    * At least one of $Q_\delta^3(z)$ and $\tilde{Q}_\delta^2(z)$ happens by self-duality and $Q_\delta^3(z) \cap \tilde{Q}_\delta^2(z) = \emptyset$.
    * This is 'regular' by RSW (i.e. we can exchange limit $z \to A_1$ and $\delta \to 0$).
  - As $z \in A_1$ moves from a_2 to a_3, $H^3(z)$ increases from 0 to 1.
    * Let $z, \tilde{z} \in A_1$ with z closer to a_2 than $\tilde{z}$. We have that $Q_\delta^3(z) \subset Q_\delta^3(\tilde{z})$, so let $R_\delta := Q_\delta^3(\tilde{z}) \setminus Q_\delta^3(z) = \{[z\tilde{z}] \leftrightsquigarrow_{\mathrm{b}} A_2\}$.
    * By RSW, the probability of the latter even is strictly positive, so $H_\delta^3(\tilde{z}) - H_\delta^3(z) = Q_\delta^3(\tilde{z}) \setminus Q_\delta^3(z)$ is uniformly positive as $\delta \to 0$.
    * By RSW, making concetric annuli, we see that $H_\delta^3(a_3) \to 1$ as $\delta \to 0$ (we can make concetric annuli around a_3).

6.11. **Discrete Cauchy-Riemann equations.**

- This is the key identity to prove analyticity (which is itself the key property).
- For an oriented edge $\vec{e} \in \Omega_\delta$ from vertex $x \in \Omega_\delta$ to vertex $y \in \Omega_\delta$ and a function $f : \Omega_\delta \to \mathbb{C}$ we define the discrete derivative $\partial_{\vec{e}} f$ by $f(y) - f(x)$.
- For the functions H_δ^μ, we can write $\partial_{\vec{e}} H_\delta^\mu = \partial_{\vec{e}}^+ H_\delta^\mu - \partial_{\vec{e}}^- H_\delta^\mu$, where $\partial_{\vec{e}}^+ H_\delta^\mu = \mathbb{P}_{\Omega_\delta}(Q_\delta^\mu(y) \setminus Q_\delta^\mu(x))$ and $\partial_{\vec{e}}^- H_\delta^\mu = \mathbb{P}_{\Omega_\delta}(Q_\delta^\mu(x) \setminus Q_\delta^\mu(y))$.
- Set $\tau = e^{2\pi i/3}$, write $\tau\vec{e}$ for the rotation of $\vec{e}$ around its origin x by $2\pi/3$ and set $H_\delta^1, H_\delta^\tau, H_\delta^{\tau^2} := H_\delta^1, H_\delta^2, H_\delta^3$.
- The discrete Cauchy-Riemann equation: for $\mu \in \left\{1, \tau, \tau^2\right\}$ and an oriented edge $\vec{e}$, we have $\partial_{\vec{e}}^+ H_\delta^\mu = \partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu} = \partial_{\tau^2\vec{e}}^+ H_\delta^{\tau^2\mu}$.
- Proof: suppose $\mu = 1$, $\vec{e}$ an horizontal edge from left to right, let $z, w \in \Omega_\delta$ be the destinations of $\tau\vec{e}$ and $\tau^2\vec{e}$, and let us prove the first identity (everything is symmetric)
  - We have that $\partial_{\vec{e}}^+ H_\delta^\mu$ is the probability of $Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)$: this event means that there is white path $\gamma : A_2 \leftrightsquigarrow A_3$ passing between x and y and that there is a black path $\lambda : A_1 \leftrightsquigarrow \{x, z, w\}$.
  - We have that $\partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu}$ is the probability of $Q_\delta^{\tau\mu}(z) \setminus Q_\delta^{\tau\mu}(x)$: this event means that there is white path $\tilde{\gamma} : A_1 \leftrightsquigarrow A_3$ passing between x and z a black path $\tilde{\lambda} : A_2 \leftrightsquigarrow \{x, w, y\}$.
  - We construct a bijection between the $\omega \in Q_\delta^\mu(y) \setminus Q_\delta^\mu(x)$ and the $\tilde{\omega} \in Q_\delta^{\tau\mu}(z) \setminus Q_\delta^{\tau\mu}(x)$: because each configuration has the same probability, this will prove the identiy:
    * Let γ_2 be the cw-most white path from $\vec{e}$ to A_2, let λ be the ccw-most black path from $\vec{e}$ to A_1 and let γ_3 be a the part of γ that goes from e to A_3.
    * Flip the color of all the hexagons that on the ccw side of γ_2 and on the cw side of λ: the black path γ_3 becomes white, and this map is clearly invertible.
    * Flip the color of all the hexagons: λ and γ_3 become white, γ_2 becomes black and we get a configuration $\tilde{\omega} \in Q_\delta^{\tau\mu}(z) \setminus Q_\delta^{\tau\mu}(x)$.
  - Hence, we have constructed a bijection, and $\partial_{\vec{e}}^+ H_\delta^\mu = \partial_{\tau\vec{e}}^+ H_\delta^{\tau\mu}$.

6.12. **Analyticity.**

- To show analyticity, we use Morera's criterion: a continuous function $f : \Omega \to \mathbb{C}$ is holomorphic if $\oint_\gamma f(z)\,\mathrm{d}z = 0$ for any smooth closed contour γ (of course, the converse is true as well).
- Why Morera's criterion holds: we can define the antiderivative $F(z) := \int_w^z f(\zeta)\,\mathrm{d}\zeta$ (i.e. the definition is independent of the contour) and see that $F(z)$ is holomorphic.
- We will show that $\psi := H_1 + \tau H_\tau + \tau^2 H_{\tau^2}$ and $\sigma := H_1 + H_\tau + H_{\tau^2}$ (notice $\varphi = \frac{1}{3}\sigma + \frac{2}{3}\psi$) are analytic using Morera's criterion, by computing Riemann sums on lattice-level and seeing that they tend to 0.
- Let $\gamma \subset \Omega$ be smooth and let γ_δ be a honeycomb discretization of γ that uses $\mathcal{O}\left(\delta^{-1}\right)$ edges, oriented in ccw direction. Let us call U_δ the interior of γ_δ.
- Discretize $\oint_\gamma \psi(z)\,\mathrm{d}z$ by $I_\delta(\gamma, \psi) := \sum_{\vec{e} \in U_\delta} \psi_\delta(\vec{e}) \cdot \vec{e}$, where we write $\psi_\delta(\vec{e}) := \frac{1}{2}(\psi_\delta(y) + \psi_\delta(x))$ and $\vec{e} := (y - x)$ (we will identify oriented edges with complex numbers).
- We can write $\sum_{\vec{e} \in \gamma_\delta} \psi_\delta(\vec{e}) \cdot \vec{e} = \sum_{f \in \mathcal{F}_{U_\delta}} \sum_{\vec{e} \in \partial f} \psi_\delta(\vec{e}) \cdot \vec{e}$, where $\mathcal{F}_{\Omega_\delta}$ is the set of hexagonal faces of Ω_δ and ∂f is the boundary of f, oriented ccw.
- We can rewrite $\sum_{\vec{e} \in \partial f} \psi_\delta(\vec{e}) \cdot \vec{e} = -\sum_{\vec{e} \in \partial f} \partial_{\vec{e}} \psi_\delta m(\vec{e})$ by discrete resummation, where $m(\vec{e})$ is the midpoint of $\vec{e}$.
- Writing $\vec{e}^*$ for the oriented edge of the dual of Ω_δ that crosses $\vec{e}$ oriented such that $\vec{e}^*/(i\vec{e}) > 0$, and introducing $c(f)$, the center of f, we get $I_\delta(\gamma, \varphi) = \frac{1}{2} \sum_{\vec{e} \in \partial f} \partial_{\vec{e}} \psi \cdot \vec{e}^*$.

- Resumming over all the edges $\vec{e} \in \vec{\mathcal{E}}_{U_\delta}$ (taking each edge in its two possible orientations), we get $I_\delta(\gamma, \psi) = \sum_{\vec{e} \in \vec{\mathcal{E}}_{U_\delta}} \vec{e}^* \partial_{\vec{e}} \psi_\delta$ + boundary terms.
- The boundary terms tend to 0 as $\delta \to 0$: there are $\mathcal{O}\left(\frac{1}{\delta}\right)$ of them, and they are of order $o(\delta)$ (the δ comes from the edge length, the $o(1)$ from $\partial_{\vec{e}} \psi$, by RSW).
- We get

$$\frac{1}{2}\left(\sum_{\vec{e} \in \vec{\mathcal{E}}_{U_\delta}} \vec{e}^* \left(\partial_{\vec{e}}^+ H_\delta^1 + \tau \partial_{\vec{e}}^+ H_\delta^\tau + \tau^2 \partial_{\vec{e}}^+ H_\delta^{\tau^2}\right) - \sum_{\vec{e} \in \vec{\mathcal{E}}_{U_\delta}} \vec{e}^* \left(\partial_{\vec{e}}^- H_\delta^1 + \tau \partial_{\vec{e}}^- H_\delta^\tau + \tau^2 \partial_{\vec{e}}^- H_\delta^{\tau^2}\right)\right),$$

- We can resum over all the edges, to get $\sum_{\vec{e} \in \vec{\mathcal{E}}_{U_\delta}} \vec{e}^* \left(\partial_{\vec{e}}^+ H_\delta^1 + \tau \partial_{\vec{e}}^+ H_\delta^\tau + \tau^2 \partial_{\vec{e}}^+ H_\delta^{\tau^2}\right)$
- We can resum once more to get

$$\sum_{\vec{e} \in \vec{\mathcal{E}}_{U_\delta}} \vec{e}^* \left(\partial_{\vec{e}}^+ H_\delta^1 + \partial_{\tau^2 \vec{e}}^+ H_\delta^\tau + \partial_{\tau \vec{e}}^+ H_\delta^{\tau^2}\right)$$

and resum one last time

$$\sum_{\vec{e} \in \vec{\mathcal{E}}_{U_\delta}} \left(\vec{e}^* + \tau (\tau \vec{e})^* + \tau^2 \left(\tau^2 \vec{e}\right)^*\right) \partial_{\vec{e}} H_\delta^1$$

.
- This last term equals 0, as $\vec{e}^* + \tau (\tau \vec{e})^* + \tau^2 \left(\tau^2 \vec{e}\right)^* = 0$, so $\lim_{\delta \to 0} I_\delta(\gamma, \psi) = 0$, and hence ψ is analytic.
- Similarly for $\sigma := H_1 + H_\tau + H_{\tau^2}$, we get a cancellation because $\vec{e}^* + (\tau \vec{e})^* + \left(\tau^2 \vec{e}\right)^* = 0$ and σ is analytic as well.

7. Ising Model

7.1. Definition.

- The Ising model is a random assignment of ± 1 spins to the vertices of a graph G with vertices V and edges E, that interact via the edges.
- The probability of a spin configuration $(\sigma_x)_{x \in V}$ is proportional to $e^{-\beta H(\sigma)}$, where $H(\sigma) = -\sum_{\langle ij \rangle \in E} \sigma_i \sigma_j$ is the energy (the sum is over all unoriented edges) and $\beta > 0$ is the inverse temperature.
- In other words $\mathbb{P}\{\sigma\} = e^{-\beta H(\sigma)}/Z_\beta$, where $Z_\beta = \sum_{\tilde{\sigma}} e^{-\beta H(\tilde{\sigma})}$ is the partition function.
- We can introduce an external magnetic field (add $h \sum_i \sigma_i$ to $\beta H(\sigma)$), replace $\sigma_i \sigma_j$ by general couplings $J_{ij} \sigma_i \sigma_j$, ..., but we will not do that (not that it is not interesting, but things become difficult...)
- In this class, we will only consider square-grid discretizations of planar domains as graphs, but other graphs such has honeycomb domains, would work almost as well.

7.2. Motivation.

- The Ising model was introduced as a model of ferromagnetism: one would like to say that the matter of small magnets that tend to align (in ferromagnetic material).
- Explaining clearly why this happens for a number of materials (such as iron) requires quantum mechanics and is difficult.
- Anyway, the model is a strong simplification of reality (spins only take two directions, we don't have magnetic field and this is not a quantum model).
- Nevertheless, there is a strong support for universality
- General postulate in statistical mechanics (can be justified, but subtle): the probability of a configuration (in the so-called canonical ensemble) is proportional to its Boltzmann weight $\exp(-\beta H)$
- A simple justification can be given by the fact that it is the stationary measure of a natural dynamics: this is a natural way to do simulations.

7.3. Simulation.

- Metropolis algorithm/Glauber dynamics: start from an arbitrary configuration, and make random flips:
  - Compute the energy of the current configuration H_σ.
  - Pick a vertex x at random, consider the configuration ρ, obtained by flipping the spin x of σ, and compute its energy H_ρ
  - If $H_\rho \leq H_\sigma$, replace σ by ρ. If $H_\rho > H_\sigma$, replace σ by ρ with probability $e^{-\beta H_\rho}/e^{-\beta H_\sigma}$ (i.e. the relative probabilities of ρ and σ)

- This defines a Markov chain on the state space $\mathcal{S} := \{\pm 1\}^G$, with transition matrix $P_M = (P_M)_{\rho\sigma}$.

- Heat bath dynamics: start from an arbitrary configuration, and make random flips
  - Compute the energy of the current configuration
  - Pick a vertex x at random, and sample the spin σ_x at random by giving probability

$$\mathbb{P}\{\sigma_x\} = \frac{e^{-\beta H[\sigma^+]}}{e^{-\beta H[\sigma^+]} + e^{-\beta H[\sigma^-]}},$$

    where σ^+ and σ^- denote the configuration σ, with the spin σ_x forced to be $+1$ and -1 respectively.
  - This define a Markov chain on $\mathcal{S}$, with transition matrix P_H.
- Convergence: for any initial probability measure μ on $\mathcal{S}$, we have that $\mu P_M^n \to \mu_{\text{Ising}}$ and $\mu P_H^n \to \mu_{\text{Ising}}$ as $n \to \infty$.
- Markov chain statement: if P is the transition matrix of an irreducible aperiodic Markov chain (i.e. there exists N such that $\left(P^N\right)_{\sigma\rho} > 0$ for all $\sigma, \rho \in \mathcal{S}$), then $\mu P^n \to \mu_{\text{stat}}$ as $n \to \infty$, where μ_{stat} is the unique stationary measure, i.e. $\mu_{\text{stat}} P = \mu_{\text{stat}}$.
  - We know 1 is an eigenvalue of P, and hence of P^T; we know the eigenvalues of P are ≤ 1 in modulus, so are the ones of P^T.
  - Perron-Frobenius theorem: let Q be a matrix with positive entries. Then the largest eigenvalue (in modulus) is real, simple and the corresponding eigenvector has positive entries.
  - Apply P-F to $Q = P^N$, and deduce that there is a unique positive eigenvector of eigenvalue 1 for Q, but then it implies the result of P^T (otherwise, there would be a contradiction).
- Sufficient (non-necessary criterion for measure invariance): detailed balance.
  - A measure μ such that $\mu_\sigma P_{\sigma\rho} = \mu_\rho P_{\rho\sigma}$ for all $\sigma, \rho \in \mathcal{S}$ is said to satisfy detailed balance: 'flux' of probability from σ to ρ equals 'flux' of probability of ρ to σ.
  - Summing over σ, we get $(\mu P)_\rho = \mu_\rho$, i.e. μ is invariant.
  - Why is this satisfied for the Metropolis dynamics: if $H_\rho > H_\rho$, we have a flux $\frac{1}{|V|}\left(e^{-\beta H_\rho}/e^{-\beta H_\sigma}\right)e^{-\beta H_\sigma} = e^{-\beta H_\rho}$ from σ to ρ, and a flux $\frac{1}{|V|}1 \cdot e^{-\beta H_\rho}$from ρ to σ, so both are equal (the other situation is symmetric) – the $\frac{1}{|V|}$ is there because we pick the vertices uniformly.
  - Why is this satisfied for the heat-bath dynamics: if σ^-, σ^+ are configurations coinciding except at a vertex x, $\sigma_x^- = -1$ and $\sigma_x^+ = 1$, we have a flux $\frac{1}{|V|}\frac{e^{-\beta H[\sigma^-]}e^{-\beta H[\sigma^+]}}{e^{-\beta H[\sigma^+]}+e^{-\beta H[\sigma^-]}}$ from σ^- to σ^+ and flux $\frac{1}{|V|}\frac{e^{-\beta H[\sigma^+]}e^{-\beta H[\sigma^-]}}{e^{-\beta H[\sigma^+]}+e^{-\beta H[\sigma^-]}}$ from σ^+ to σ^-, which are both equal.

7.4. Consequence: Monotonicity with Respect to Boundary Conditions.

- We can couple Ising models with arbitrary boundary conditions $\mathfrak{b}$, with + b.c. and with − b.c. such that $\sigma^{(-)} \leq \sigma^{(\mathfrak{b})} \leq \sigma^{(+)}$ (where $\sigma \leq \rho$ means $\sigma_x \leq \rho_x$ for all $x \in G$)
  - Proof: we can couple the heat-bath dynamics $\sigma_n^{(+)}$, $\sigma_n^{(-)}$ with + and − boundary conditions (we do not flip the boundary spins), together with $\sigma_n^{(\mathfrak{b})}$ such that at any time $n \geq 0$, $\sigma_n^{(-)} \leq \sigma_n^{(\mathfrak{b})} \leq \sigma_n^{(+)}$
  - Passing to $n \to \infty$, we get the result for the spins.

7.5. Low-temperature expansion.

- Consider the Ising model on the dual Ω_δ^* (i.e. we put spins on the faces), with + boundary conditions.
- For each spin configuration $\sigma \in \{\pm 1\}^{\Omega_\delta^*}$, we put an edge of Ω_δ for any pair of spins that are different: we get a collection of closed loops on Ω_δ and this is actually a bijection.
- For a collection of closed loops ω on Ω_δ, the energy of the corresponding spin configuration equals $|\omega| - (\#\text{edges}(\Omega_\delta) - |\omega|) = 2|\omega| - (\#\text{edges}(\Omega_\delta))$.
- We can hence rewrite the partition function of the Ising model as $e^{\beta\#\text{edges}(\Omega_\delta)}\sum_{\omega\in\mathcal{C}(\Omega_\delta)} e^{-2\beta|\omega|}$, where $\mathcal{C}(\Omega_\delta)$ is the set of collection of edges of Ω_δ that form closed loops.
- The sign of a spin becomes the parity of the number of loops that surround it.

7.6. High-temperature expansion.

- Consider the Ising model on Ω_δ, with no boundary conditions (free boundary conditions). Let us expand the partition function of the Ising model

$$\begin{aligned}
&\sum_{\sigma\in\mathcal{S}} e^{-\beta H(\sigma)} && = \sum_{\sigma\in\mathcal{S}}\prod_{\langle xy\rangle} e^{\beta\sigma_x\sigma_y} \\
&= \sum_{\sigma\in\mathcal{S}}\prod_{\langle xy\rangle}\left(\cosh\left(\beta\sigma_x\sigma_y\right)+\sinh\left(\beta\sigma_x\sigma_y\right)\right) && = \sum_{\sigma\in\mathcal{S}}\prod_{\langle xy\rangle\in E}\cosh\left(\beta\right)\left(1+\sigma_x\sigma_y\tanh\beta\right) \\
&= \left(\cosh\left(\beta\right)\right)^{\#E}\sum_{\sigma\in\mathcal{S}}\sum_{\mathcal{E}\subset E}\left(\tanh\beta\right)^{\mathcal{E}}\prod_{\langle xy\rangle\in\mathcal{E}}\sigma_x\sigma_y && = \left(\cosh\left(\beta\right)\right)^{\#E}\sum_{\mathcal{E}\subset E}\left(\tanh\beta\right)^{\mathcal{E}}\sum_{\sigma\in\mathcal{S}}\prod_{\langle xy\rangle\in\mathcal{E}}\sigma_x\sigma_y
\end{aligned}$$

- We have that $\sum_{\sigma\in\mathcal{S}}\prod_{\langle xy\rangle\in\mathcal{E}}\sigma_x\sigma_y$ is zero, $\partial\mathcal{E}=\emptyset$, where $\partial\mathcal{E}$ is the set of vertices that have odd degree in $\mathcal{E}$.
  - If there is a vertex $x\in\Omega_\delta$ that appears in an odd number of edges the term will be 0: half of the time, this term will appear with $+$ sign, half of the time with $-$ sign and hence terms will cancel.
  - By Euler's theorem for walks, $\mathcal{E}$ must be a collection of loops.
  - Otherwise, we get $2^{\#V}$ terms equal to $+1$ each.

7.7. **Boundary conditions and spin correlations.**

- If we want to compute the spin correlations in the high-temperature extension with $+$ boundary conditions, we can do the same computation, except that we are not summing over the spins of $\partial\Omega$.
- The condition for a contour to have a nonzero contribution is $\partial\mathcal{E}\subset\partial\Omega$: they are the contours we would get from low-temperature expansion of Ising model on Ω^* with free boundary conditions.
- Now, with free boundary conditions, if we want to compute $\mathbb{E}^{\text{free}}_{\Omega_\delta}\left[\sigma_a\sigma_b\right]$

$$\frac{\sum_{\sigma\in\mathcal{S}}\sigma_a\sigma_b e^{-\beta H(\sigma)}}{\sum_{\sigma\in\mathcal{S}}e^{-\beta H(\sigma)}}=\frac{\sum_{\mathcal{E}\subset E:\partial\mathcal{E}=\{a,b\}}\left(\tanh\beta\right)^{\mathcal{E}}}{\sum_{\mathcal{E}\subset E:\partial\mathcal{E}=\emptyset}\left(\tanh\beta\right)^{\mathcal{E}}},$$

  (the additional insertion of spins at a,b changes the parity rule at these vertices).
- If a,b are put on the boundary, it we get the low-temperature expansion of an Ising model with $+$ boundary conditions on $[ab]$ and $-$ boundary conditions on $[ba]$.
- Now, if we compute $\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]$, we get in the numerator a sum over diagrams $\mathcal{E}$ such that $a\in\partial\mathcal{E}$ and $\mathcal{E}\setminus\{a\}\subset\partial\Omega$: a path from a to $\partial\Omega$, plus arcs between points of $\partial\Omega$.

7.8. **Kramers-Wannier duality.**

- Hence we get $2^{\#V}\cosh\left(\beta\right)^{\#E}\sum_{\mathcal{E}:\text{loops}}\left(\tanh\beta\right)^{\mathcal{E}}$: the terms in the sum are the same as in the low-temperature expansion of the Ising model on Ω^*_δ: so, we have exchanged
  - The graph $\Omega^*_\delta\leftrightarrow\Omega_\delta$ and its dual
  - The boundary conditions $+\leftrightarrow$ free
  - Boundary spin operators and boundary change operators
  - The parameter $e^{-2\beta}\leftrightarrow\tanh\beta$. Let's call β^* the parameter such that $\tanh\left(\beta^*\right)=e^{-2\beta}$, i.e. $\sinh\left(\beta\right)\sinh\left(\beta^*\right)=1$: if β increases, β^* decreases.
- The critical point is the self-dual point $\beta=\beta^*$, i.e. when $\beta=\frac{1}{2}\ln\left(\sqrt{2}+1\right)$.

7.9. **Peierls argument.**

- Why can we choose a $\beta>0$ sufficiently large so that $\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]\geq C>0$ uniformly with respect to δ.
- We look at loops surrounding a in low-temperature expansion. If we can show that the probability that there is no loop surrounding a is $>\frac{1}{2}$, then we are done.
- The number of loops of length ℓ surrounding 0 is $\leq\ell 4^{\ell}$ (the loop must start at distance less than ℓ to the left of a, say, and after that it is a path).
- The contributions of the configurations with a loop surrounding a is hence $\leq\sum_\ell\ell 4^\ell e^{-2\beta\ell}$, so choosing β large enough, this will be less than $1/2$.

7.10. **Dual Peierls argument.**

- Why can we choose $\beta>0$ small enough so that $\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]\to 0$?
- We can evaluate $\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]$ using the high-temperature expansion and show that $\mathbb{E}^{+}_{\Omega_\delta}\left[\sigma_a\right]\leq\sum_{\pi:a\to\partial\Omega_\delta}\tanh^{\#\pi}\left(\beta\right)$, where the sum is over all connected sets of edges connecting a and $\partial\Omega_\delta$, such that each vertex $v\notin\{a\}\cup\partial\Omega_\delta$ has even degree

- Proof: we have

$$\begin{aligned}\mathbb{E}^{+}_{\Omega_\delta}[\sigma_a] &= \frac{\sum_{\mathcal{E}\subset E: a\in\partial E, \partial E\setminus\{a\}\subset\partial\Omega_\delta}\tanh^{\#\mathcal{E}}(\beta)}{\sum_{\mathcal{E}\subset E:\partial E\subset\partial\Omega_\delta}\tanh^{\#\mathcal{E}}(\beta)} \\ &= \sum_{\pi: a\to\partial\Omega_\delta}\tanh^{\#\pi}(\beta)\frac{\sum_{\mathcal{E}\subset E\setminus\pi:\partial\mathcal{E}\subset\partial\Omega_\delta\setminus\pi}\tanh^{\#\mathcal{E}}(\beta)}{\sum_{\mathcal{E}\subset E:\partial E\subset\partial\Omega_\delta}\tanh^{\#\mathcal{E}}(\beta)} \\ &\leq \sum_{\pi: a\to\partial\Omega_\delta}\tanh^{\#\pi}(\beta)\end{aligned}$$

- Now, we have $\#\{\pi : a\to\partial\Omega_\delta : |\pi|\leq\ell\}\leq 4^\ell$ for each connected graph with ℓ vertices, we can find a path that passes through each edge exactly twice, and there at most $4^{2\ell}$ paths of length 2ℓ
- So, if choose β small enough this goes to 0 as $\delta\to 0$.

7.11. Monotonicity with Respect to β [to be completed].

- Another monotonicity property for the Ising model is that of the magnetization with respect to the temperature: if we consider the Ising model with + boundary conditions, then $\beta\mapsto\mathbb{E}^{+}_{G;\beta}[\sigma_x]$ is an increasing function of β.
- The proof of this relies on the FKG inequality for the FK representation of the Ising model.
- The FK model consists of a collection of random clusters

8. Dimer Model

8.1. Generalities about the dimer model.

- We consider random dimer tiling (i.e. domino covering or perfect matching) of a given graph.
- This model exhibits an extremely rich behavior and many more 'physical' models can be mapped to it.
- The dimer model is very well understood on *bipartite (i.e. 2-colorable) planar graphs*.
- This lecture is focused on the dimer model on square grid domains (subgraphs of $\mathbb{Z}^2$).

8.2. Number of domino tilings of a checkerboard: statement. Today, we want to prove a classical theorem about dimer counting [Kas61, FiTe61]:

- The number of domino tilings of an $p\times q$ checkerboard is given by (unless pq is odd)

$$\sqrt{\prod_{j=1}^{p}\prod_{k=1}^{q}\left(2\cos\left(\frac{\pi j}{p+1}\right)+2i\cos\left(\frac{\pi k}{q+1}\right)\right)}.$$

- The proof of this result will teach us interesting things about the dimer model.

8.3. Key steps of the proof: Two main parts:

- Write the number of domino tilings in terms of the determinant of a matrix.
- Diagonalize the matrix and take the product of the eigenvalues.

8.4. Preliminary result: number of domino tilings as a permanent.

- Consider a square grid domain G with black vertices $\mathcal{B}=\{b_1,\ldots,b_n\}$ and white vertices $\mathcal{W}=\{w_1,\ldots,w_n\}$ (we color the vertices in such a way that adjacent vertices are of different colors).
- (Reduced) adjacency matrix A of G: an $n\times n$ matrix $(a_{bw})_{b\in\mathcal{B},w\in\mathcal{W}}$ indexed by the black/white vertices of G such that $a_{bw}=1$ if $b\sim w$ and $a_{bw}=0$ otherwise.
- Permanent of A: $\mathrm{Per}(A)=\sum_{\sigma\in\mathcal{S}_n}a_{b_1w_{\sigma(1)}}\cdots a_{b_nw_{\sigma(n)}}$ (like the determinant without the signature)
- In the sum $\sum_{\sigma\in\mathcal{S}_n}a_{b_1w_{\sigma(1)}}\cdots a_{b_nw_{\sigma(n)}}$, we get a nonzero term whenever $b_1\sim w_{\sigma(1)},\ldots,b_n\sim w_{\sigma(n)}$, i.e. $\langle b_1w_{\sigma(1)}\rangle$, ..., $\langle b_nw_{\sigma(n)}\rangle$ is a dimer cover.
- So: $\mathrm{Per}(A)=\#$dimer covers (this works for any bipartite graph),
- Inconvenient: this not very useful. Unlike the determinant, the permanent is very hard to compute (NP hard) and does not have good properties.

8.5. Next step: number of domino tilings as a determinant.

- Kasteleyn matrix $K = (k_{bw})_{b\in\mathcal{B},w\in\mathcal{W}}$ for the square grid (can be generalized to other planar graphs):
  - $k_{bw} = 1$ if $\langle bw\rangle$ is a horizontal edge,
  - $k_{bw} = i = \sqrt{-1}$ if $\langle bw\rangle$ is a vertical edge and
  - $k_{bw} = 0$ otherwise.
- Result: $|\det K| = \#$dimer covers of G.
  - Proof the result, we expand:

$$\det K = \sum_{\sigma\in\mathcal{S}_n} \epsilon(\sigma)\, k_{b_1 w_{\sigma(1)}} \cdots k_{b_n w_{\sigma(n)}}.$$

  - As before, all nonzero terms correspond to a dimer tiling. To get the result, enough to show the following lemma:
- Lemma: Let σ and $\tilde{\sigma}$ be two permutation corresponding to nonzero terms. Then

$$\epsilon(\sigma)\, k_{b_1 w_{\sigma(1)}} \cdots k_{b_n w_{\sigma(n)}} = \epsilon(\tilde{\sigma})\, k_{b_1 w_{\tilde{\sigma}(1)}} \cdots k_{b_n w_{\tilde{\sigma}(n)}}.$$

  - Let $\mathcal{T}$ and $\tilde{\mathcal{T}}$ be the corresponding dimer tilers. If we superpose $\mathcal{T}$ and $\tilde{\mathcal{T}}$ (i.e. take the XOR $\mathcal{T}\oplus\tilde{\mathcal{T}}$). we get a collection of loops on G.
  - We can move from $\mathcal{T}$ to $\tilde{\mathcal{T}}$ by 'rotating' the dimers of $\mathcal{T}$ along each loop.
  - We can suppose that $\mathcal{T}\oplus\tilde{\mathcal{T}}$ just consists of one loop $b_{i_1}w_{i_1}\cdots b_{i_k}w_{i_k}$ and that

$$\langle b_{i_1}w_{i_1}\rangle, \ldots, \langle b_{i_k}w_{i_k}\rangle \in \mathcal{T} \quad\text{and}\quad \langle b_{i_1}w_{i_k}\rangle, \ldots, \langle b_{i_k}w_{i_1}\rangle \in \tilde{\mathcal{T}}.$$

  - We have that $\sigma = (i_1\ \ldots\ i_k)\circ\tilde{\sigma}$ and $\epsilon(\sigma) = (-1)^{k+1}\epsilon(\tilde{\sigma})$.
  - To get the desired result

$$\epsilon(\sigma)\, k_{b_1 w_{\sigma(1)}} \cdots k_{b_n w_{\sigma(n)}} = \epsilon(\tilde{\sigma})\, k_{b_1 w_{\tilde{\sigma}(1)}} \cdots k_{b_n w_{\tilde{\sigma}(n)}}$$

  one should check that

$$k_{b_1 w_{\sigma(1)}} \cdots k_{b_n w_{\sigma(n)}} = (-1)^{k+1} k_{b_1 w_{\tilde{\sigma}(1)}} \cdots k_{b_n w_{\tilde{\sigma}(n)}}$$

  .
  - Provided by the following lemma, using that the number of vertices inside the loop $b_{i_1}w_{i_1}\cdots b_{i_k}w_{i_k}$ is even (it can be tiled by dimers):
- Lemma: for any cycle $b_1w_1\ldots b_kw_k$, if $m_1 = k_{b_1w_1}b_{b_2w_2}\ldots k_{b_kw_k}$ and $m_2 = k_{b_2w_1}k_{b_3w_2}\ldots k_{b_1w_k}$, we have that $m_1 = (-1)^{\ell+k+1} m_2$, where ℓ is the number of vertices strictly inside the cycle.
  - Proof: by induction (check that when one adds a face to the domain inside the cycle, the property is maintained).

8.6. Computing the determinant.

- Now that we have $|\det K| = \#$dimer tilings, how to get the formula for the number of tilings of an $p\times q$ checkerboard, with $2n$ vertices?
- Consider the (extended) Kasteleyn matrix: it is the $2n\times 2n$ matrix $\tilde{K} = \begin{pmatrix} 0 & K \\ K^T & 0\end{pmatrix}$.
  - $\tilde{K}$ is simply the matrix indexed by $\{b_1,\ldots,b_n,w_1,\ldots,w_n\}$ (in that order) such that $k_{vw} = 1$ if $\langle vw\rangle$ is a horizontal edge, $k_{vw} = i$ is $\langle vw\rangle$ is vertical and $k_{vw} = 0$ otherwise.
  - We have that $\left|\det\tilde{K}\right| = |\det K|^2$ (easy to see from the expansion of the determinant)
- It remains to show that:

$$\det\tilde{K} = \prod_{j=1}^{p}\prod_{k=1}^{q}\left(2\cos\left(\frac{\pi j}{p+1}\right) + 2i\cos\left(\frac{\pi k}{q+1}\right)\right)$$

- How to show that?
  - Let us find $p\times q$ independent eigenvectors of $\tilde{K}$ and compute their eigenvalues.
  - Identify the vectors indexed by the vertices with *functions* defined on the checkerboard $\{1,\ldots,p\}\times\{1,\ldots,q\}$. We have

$$\left(\tilde{K}f\right)(x,y) = f(x+1,y) + f(x-1,y) + i\left(f(x,y+1) + f(x,y-1)\right),$$

  where f is set to 0 when taken outside of the range.

– For $j \in \{1, \ldots, p\}$ and $k \in \{1, \ldots, q\}$, let $z := e^{\frac{i\pi j}{p+1}}$ and $w := e^{\frac{i\pi k}{q+1}}$ and consider the vector

$$f^{jk}(x, y) := \left(z^x - z^{-x}\right)\left(w^y - w^{-y}\right) = -4 \sin\left(\frac{\pi j x}{p+1}\right) \sin\left(\frac{\pi k y}{q+1}\right),$$

notice that f^{jk} is zero when $x = 0, p+1$ or $y = 0, q+1$.

– We have that $f^{jk}_{(x,y)}$ is an eigenvector of $\tilde{K}$, of eigenvalue $\lambda = z + \frac{1}{z} + i\left(w + \frac{1}{w}\right)$: one can see that

$$\left(\tilde{K} f\right)(x, y) = \lambda f^{jk}(x, y)$$

(we set $f^{jk}(x, y) = 0$ if either x or y go outside of the checker board)

– Hence $\det \tilde{K}$ is the product of the eigenvalues (the eigenvectors are independent, as they have distinct eigenvalues).