For exercises 1, 2 and 3, we consider the Ising model with + boundary conditions on the square lattice inside the open unit disc $\mathbb{D} \subset \mathbb{R}^2$. We denote by $\mathbb{D}_\delta$ the discretisation $\mathbb{D} \cap \delta\mathbb{Z}^2$.

**Exercise 1.** *Coupling and stochastic domination*

(1) Recall the Markov Chain for the Ising model that you have seen in class (the Glauber dynamics).

**Solution.** The Markov Chain you have seen consists of the following steps:
(a) Start from an arbitrary configuration,
(b) Make random flips:
(i) Compute the energy of the current configuration $H_\sigma$.
(ii) Pick a vertex $x$ at random, consider the configuration $\rho$ obtained by flipping the spin $x$ of $\sigma$, and compute its energy $H_\rho$
(iii) If $H_\rho \leq H_\sigma$, replace $\sigma$ by $\rho$. If $H_\rho > H_\sigma$, replace $\sigma$ by $\rho$ with probability $e^{-\beta H_\rho}/e^{-\beta H_\sigma}$ .

(2) Consider the following Heat Bath Dynamics :
(a) Pick a vertex $x$ at random,
(b) Sample the spin $\sigma_x$ at random by giving probability

$$\mathbb{P}(\sigma_x = 1) = \frac{e^{-\beta\mathcal{H}(\sigma^+)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}$$

where $\sigma^+$ and $\sigma^-$ denote the configuration $\sigma$ with the spin $\sigma_x$ forced to be $+1$ and $-1$ respectively. Prove that the Ising measure is the invariant probability measure of this dynamics. *Hint: check the detailed balance equation.*

**Solution.** We will prove the detailed balance equation :

$$\pi_{Ising}(\sigma) P_{HeatBath}(\sigma, \rho) = \pi_{Ising}(\rho) P_{HeatBath}(\rho, \sigma).$$

If $\rho$ is not of the form $\sigma^+$ or $\sigma^-$, the detailed balance equation is trivially true since $P_{HeatBath}(\sigma, \rho) = P_{HeatBath}(\rho, \sigma) = 0$. Now, let us suppose there exists a vertex $x$ such that $\rho = \sigma^+$ , then

$$\pi_{Ising}(\sigma) P_{HeatBath}(\sigma, \rho) = \pi_{Ising}(\sigma) P_{HeatBath}(\sigma, \sigma^+) = \frac{e^{-\beta\mathcal{H}(\sigma)}}{Z} \frac{e^{-\beta\mathcal{H}(\sigma^+)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}$$

and

$$\pi_{Ising}(\rho) P_{HeatBath}(\rho, \sigma) = \pi_{Ising}(\sigma^+) P_{HeatBath}(\sigma^+, \sigma) = \frac{e^{-\beta\mathcal{H}(\sigma^+)}}{Z} \frac{e^{-\beta\mathcal{H}(\sigma)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}.$$

This proves that the detailed balance equation is valid and the Ising measure is the invariant probability measure of this dynamics.

(3) We define a partial ordering between spin configurations $\sigma \in \{\pm 1\}^{\mathbb{D}_\delta}$ : $\sigma \leq \sigma'$ if $\sigma_a \leq \sigma'_a$ for all $a \in \mathbb{D}_\delta$. Suppose that we start the chain at a common temperature $\beta > 0$ on two starting configurations $\sigma^0 \leq \sigma'^0$ . Show that we can couple the two dynamics such that this ordering is preserved at each step of the Markov Chain, that is

$$\sigma^n \leq \sigma'^n$$

for all the time steps $n \in \mathbb{N}$.

**Solution.** We will define two Markov Chain $\sigma^n$ and $\sigma'^n$ starting from $\sigma^0$ and $\sigma'^0$ by using the Heat Bath Dynamics and:
(a) picking the same vertex $x$ at random for the two Markov Chain,
(b) sampling the spin $\sigma_x^{n+1}$ and $\sigma'^{n+1}_x$ using the same underlying uniform random variable: we consider $U \sim Uni([0,1])$ and we define

$$\sigma_x^{n+1} = 1 \text{ if } U \leq \frac{e^{-\beta\mathcal{H}(\sigma^{n+})}}{e^{-\beta\mathcal{H}(\sigma^{n+})} + e^{-\beta\mathcal{H}(\sigma^{n-})}}$$

and $\sigma_x^{n+1} = -1$ if not,

$$\sigma'^{n+1}_x = 1 \text{ if } U \leq \frac{e^{-\beta\mathcal{H}(\sigma'^{n+})}}{e^{-\beta\mathcal{H}(\sigma'^{n+})} + e^{-\beta\mathcal{H}(\sigma'^{n-})}}$$

and $\sigma'^{n+1}_x = -1$ if not.

If we prove that at any time $\frac{e^{-\beta\mathcal{H}(\sigma^{n+})}}{e^{-\beta\mathcal{H}(\sigma^{n+})}+e^{-\beta\mathcal{H}(\sigma^{n-})}} \leq \frac{e^{-\beta\mathcal{H}(\sigma'^{n+})}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}+e^{-\beta\mathcal{H}(\sigma'^{n-})}}$ then by induction we can conclude that $\sigma^n \leq \sigma'^n$. In order to prove the first inequality, we only need to prove that

$$\frac{e^{-\beta\mathcal{H}(\sigma^{n+})}+e^{-\beta\mathcal{H}(\sigma^{n-})}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} \geq \frac{e^{-\beta\mathcal{H}(\sigma'^{n+})}+e^{-\beta\mathcal{H}(\sigma'^{n-})}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}}$$

or

$$\frac{e^{-\beta\mathcal{H}(\sigma^{n-})}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} \geq \frac{e^{-\beta\mathcal{H}(\sigma'^{n-})}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}}.$$

Let us remark that for a configuration $\sigma$ and any site $x$,

$$\frac{e^{-\beta\mathcal{H}(\sigma^{-})}}{e^{-\beta\mathcal{H}(\sigma^{+})}} = e^{\beta\left(-\sum_{a\sim b}\sigma_a^+\sigma_b^+ + \sum_{a\sim b}\sigma_a^-\sigma_b^-\right)}$$

(Be careful, the energy $\mathcal{H}$ is equal to $-\sum_{x\sim y}\sigma_x\sigma_y$. Do not forget the $-$ sign), yet $\sigma^+$ and $\sigma^-$ only differs at $x$, thus it is equal to $e^{-2\beta\sum_{a\sim x}\sigma_a}$. This implies that

$$\frac{e^{-\beta\mathcal{H}(\sigma^{n-})}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} = e^{-2\beta\sum_{a\sim x}\sigma_a^n} \geq e^{-2\beta\sum_{a\sim x}\sigma_a'^n} = \frac{e^{-\beta\mathcal{H}(\sigma'^{n-})}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}},$$

which allows us to conclude.

**Exercise 2.** *Monotonicity property for the boundary conditions*

Show that if $\mathfrak{b}_1, \mathfrak{b}_2 \in \{\pm 1\}^{\partial\mathbb{D}_\delta}$ are boundary conditions such that $\mathfrak{b}_1 \leq \mathfrak{b}_2$ (which means that for any element $x$ of the boundary $\mathfrak{b}_1(x) \leq \mathfrak{b}_2(x)$). Then the corresponding Ising measures satisfy:

$$\mathbb{E}^\beta_{\mathbb{D}_\delta;\mathfrak{b}_1}(\sigma_a) \leq \mathbb{E}^\beta_{\mathbb{D}_\delta;\mathfrak{b}_2}(\sigma_a)$$

for any $a \in \mathbb{D}_\delta$. *Hint: Use the Markov chain dynamics seen in the previous exercise; the boundary spins remain unchanged.*

**Solution.** One just has to use the same coupling of Markov chains (where we never pick $x$ on the boundary) as in the previous exercise, with similar initial condition except for the boundary where one begins with $\mathfrak{b}_1$ and $\mathfrak{b}_2$. The result follows from the general fact about Markov chains that the Glauber or Heat bath dynamics converge to the Ising measure over spin configurations.

**Exercise 3.** *Low-temperature expansion*

The aim of this exercise is to show that there exists $0 < \beta < \infty$ (large enough) such that

$$\liminf_{\delta\to 0} \mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99.$$

Let us fix $\delta$, we will show that $\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \epsilon(\beta)$ where $\epsilon(\beta) \to 0$ as $\beta \to \infty$ is a function independent of $\delta$.

(1) Verify that showing $\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \epsilon(\beta)$ is already enough to prove $\liminf_{\delta\to 0}\mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99$.

**Solution.** We have $\mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) = \mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = 1\right) - \mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) = 1 - 2\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \geq 1 - 2\epsilon(\beta)$. To get the desired estimate, it suffices to choose $\beta$ large enough such that $2\epsilon(\beta) < 0.01$.

(2) Recall the partition function of the Ising model on $\mathbb{D}_\delta$ with $+$ boundary conditions:

$$Z_{\mathbb{D}_\delta,+} = \sum_{\sigma\in\{\pm1\}^{\mathbb{D}_\delta,+}} e^{\beta\sum_{xy\in\mathcal{E}}\sigma_x\sigma_y}.$$

Using the relation $\sum_{xy\in\mathcal{E}}\sigma_x\sigma_y = |\mathcal{E}| - \sum_{xy\in\mathcal{E}}(1-\sigma_x\sigma_y)$, express the Hamiltonian and the partition function in terms of the loops of $\sigma$.

**Solution.** We have:

$$\mathcal{H}_{\mathbb{D}_\delta,+}(\sigma) = -\sum_{\{x,y\}\in\mathcal{E}}\sigma_x\sigma_y = -|\mathcal{E}| + \sum_{\{x,y\}\in\mathcal{E}} 1 - \sigma_x\sigma_y = -|\mathcal{E}| + \sum_{\{x,y\}\in\mathcal{E},\sigma_x\neq\sigma_y} 2 = -|\mathcal{E}| + 2|\{\{x,y\}\in\mathcal{E}, \sigma_x \neq \sigma_y\}|.$$

As we have seen in the lecture, each configuration $\sigma$ defines a set of loops in the dual graph of $\mathbb{D}_\delta$, each loop edge corresponds to an original edge between vertices whose spins differ; we denote this set of loops as $\mathcal{C}(\sigma)$. We can thus rewrite the last sum as:

$$H_{\mathbb{D}_\delta,+}(\sigma) = -|\mathcal{E}| + 2\sum_{\gamma\in\mathcal{C}(\sigma)} |\gamma|.$$

Thus, the partition function can be expressed as:

$$Z \quad = \quad \sum_{\sigma\in\{\pm 1\}^{\mathbb{D}_\delta,+}} e^{\beta|\mathcal{E}|-2\beta\sum_{\gamma\in\mathcal{C}(\sigma)}|\gamma|} = e^{\beta|\mathcal{E}|}\sum_{\sigma\in\{\pm 1\}^{\mathbb{D}_\delta,+}}\prod_{\gamma\in\mathcal{C}(\sigma)} e^{-2\beta|\gamma|}.$$

Notice that when we consider the probability of a particular configuration $\sigma$ using this expression, the term $e^{\beta|\mathcal{E}|}$ cancels out in the numerator and denominator, so we can disregard it.

(3) What is en equivalent way to describe the event $\sigma_{(0,0)} = -1$ in terms of the contours surrounding $(0,0)$? Conclude that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \le \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\exists\gamma^* \in \mathcal{C}(\sigma)$ a loop surrounding$(0,0))$. *Hint: What has to be the parity of the number of loops?*

**Solution.** In order for $\sigma_{(0,0)} = -1$, the vertex $(0,0)$ has to be surrounded by an odd number of loops. This in particular implies, that there needs to exist at least one loop surrounding (0,0).

(4) Let us fix a particular loop $\gamma^*$ which surrounds $(0,0)$. Show that

$$\frac{\sum_{\sigma:\gamma^*\in\mathcal{C}(\sigma)}\prod_{\gamma\in\mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|}}{\sum_{\sigma'}\prod_{\gamma'\in\mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}} \le 1.$$

**Solution.** We will show that the numerator is less than or equal to the denominator as follows: for every configuration $\sigma$ that contains the loop $\gamma^*$ we construct a new configuration $\sigma'$ such that $\prod_{\gamma\in\mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|} = \prod_{\gamma'\in\mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}$. In other words, we want $\sigma'$ to have the exact same loops as $\sigma$, just without $\gamma^*$. To construct $\sigma'$, simply take $\sigma$ and flip all the spins in the interior of the $\gamma^*$ loop; it is not hard to see that the resulting configuration $\sigma'$ will have all the loops as $\sigma$, except for $\gamma^*$. Moreover, from the construction, it is apparent that the mapping $\sigma \mapsto \sigma'$ is injective. Thus, each product $\prod_{\gamma\in\mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|}$ in the numerator is also present in the denominator as $\prod_{\gamma'\in\mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}$which concludes the proof.

(5) Show that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right)$ is bounded above by $\sum_{\ell\ge 1}\ell 4^{\ell} e^{-2\beta\ell}$.

**Solution.** We have that:

$$\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \le \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\exists\gamma^* \in \mathcal{C}(\sigma) \text{ a loop surrounding}(0,0)) \ \le \sum_{\gamma^*}\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\gamma^*) = \sum_{\gamma^*}\sum_{\sigma:\gamma^*\in\mathcal{C}(\sigma)}\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\sigma)$$

Using (2), we can rewrite the probability of $\sigma$ in terms of its contours:

$$\sum_{\gamma^*}\sum_{\sigma:\gamma^*\in\mathcal{C}(\sigma)}\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\sigma) = \sum_{\gamma^*}\frac{\sum_{\sigma:\gamma^*\in\mathcal{C}(\sigma)}\prod_{\gamma\in\mathcal{C}(\sigma)} e^{-2\beta|\gamma|}}{\sum_{\sigma'}\prod_{\gamma'\in\mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}} = \sum_{\gamma^*} e^{-2\beta|\gamma^*|}\frac{\sum_{\sigma:\gamma^*\in\mathcal{C}(\sigma)}\prod_{\gamma\in\mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|}}{\sum_{\sigma'}\prod_{\gamma'\in\mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}} \le \sum_{\gamma^*} e^{-2\beta|\gamma^*|}.$$

Where the last inequality uses point (4). Now, it remains to estimate the number of loops around $(0,0)$ with a fixed length $l$ (you have seen this result in the lecture as well as in Exercise sheet 8, exercise 3.). This gives the final inequality

$$\sum_{\gamma^*} e^{-2\beta|\gamma^*|} \le \sum_{\ell\ge 1}\ell 4^{\ell} e^{-2\beta\ell}.$$

(6) Conclude that there exists $0 < \beta < \infty$ (large enough) such that

$$\liminf_{\delta\to 0}\mathbb{E}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \ge 0.99.$$

**Solution.** Since $\sum_{\ell\ge 1}\ell 4^{\ell} e^{-2\beta\ell}$ converges to 0 as $\beta\to\infty$ we have shown that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \le \epsilon(\beta)$ where $\epsilon(\beta)\to 0$ as $\beta\to\infty$ is a function independent of $\delta$. Thus, using (1), the proof is finished.