For exercises 1, 2 and 3, we consider the Ising model with + boundary conditions on the square lattice inside the open unit disc $\mathbb{D} \subset \mathbb{R}^2$. We denote by $\mathbb{D}_\delta$ the discretisation $\mathbb{D} \cap \delta\mathbb{Z}^2$.

Exercise 1. *Coupling and stochastic domination*

(1) Recall the Markov Chain for the Ising model that you have seen in class (the Glauber dynamics).
(2) Consider the following Heat Bath Dynamics :
  (a) Pick a vertex x at random,
  (b) Sample the spin σ_x at random by giving probability

$$\mathbb{P}\left(\sigma_x = 1\right) = \frac{e^{-\beta\mathcal{H}(\sigma^+)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}$$

  where σ^+ and σ^- denote the configuration σ with the spin σ_x forced to be $+1$ and -1 respectively.

  Prove that the Ising measure is the invariant probability measure of this dynamics. *Hint: check the detailed balance equation.*
(3) We define a partial ordering between spin configurations $\sigma \in \{\pm 1\}^{\mathbb{D}_\delta}$: $\sigma \leq \sigma'$ if $\sigma_a \leq \sigma'_a$ for all $a \in \mathbb{D}_\delta$. Suppose that we start the chain at a common temperature $\beta > 0$ on two starting configurations $\sigma^0 \leq {\sigma'}^0$. Show that we can couple the two dynamics such that this ordering is preserved at each step of the Markov Chain, that is

$$\sigma^n \leq {\sigma'}^n$$

  for all the time steps $n \in \mathbb{N}$.

Exercise 2. *Monotonicity property for the boundary conditions*

Show that if $\mathfrak{b}_1, \mathfrak{b}_2 \in \{\pm 1\}^{\partial\mathbb{D}_\delta}$ are boundary conditions such that $\mathfrak{b}_1 \leq \mathfrak{b}_2$ (which means that for any element x of the boundary $\mathfrak{b}_1(x) \leq \mathfrak{b}_2(x)$). Then the corresponding Ising measures satisfy:

$$\mathbb{E}^\beta_{\mathbb{D}_\delta;\mathfrak{b}_1}\left(\sigma_a\right) \leq \mathbb{E}^\beta_{\mathbb{D}_\delta;\mathfrak{b}_2}\left(\sigma_a\right)$$

for any $a \in \mathbb{D}_\delta$. *Hint: Use the Markov chain dynamics seen in the previous exercise; the boundary spins remain unchanged.*

Exercise 3. *Low-temperature expansion*

The aim of this exercise is to show that there exists $0 < \beta < \infty$ (large enough) such that

$$\liminf_{\delta\to 0} \mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99.$$

Let us fix δ, we will show that $\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \epsilon(\beta)$ where $\epsilon(\beta) \to 0$ as $\beta \to \infty$ is a function independent of δ.

(1) Verify that showing $\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \epsilon(\beta)$ is already enough to prove $\liminf_{\delta\to 0} \mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99$.
(2) Recall the partition function of the Ising model on $\mathbb{D}_\delta$ with + boundary conditions:

$$Z_{\mathbb{D}_\delta,+} = \sum_{\sigma\in\{\pm 1\}^{\mathbb{D}_\delta,+}} e^{\beta\sum_{xy\in\mathcal{E}}\sigma_x\sigma_y}.$$

  Using the relation $\sum_{xy\in\mathcal{E}}\sigma_x\sigma_y = |\mathcal{E}| - \sum_{xy\in\mathcal{E}}(1-\sigma_x\sigma_y)$, express the Hamiltonian and the partition function in terms of the loops of σ.
(3) What is en equivalent way to describe the event $\sigma_{(0,0)} = -1$ in terms of the contours surrounding $(0,0)$? Conclude that $\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \mathbb{P}^\beta_{\mathbb{D}_\delta,+}(\exists\gamma^* \in \mathcal{C}(\sigma) \text{ a loop surrounding}(0,0))$. *Hint: What has to be the parity of the number of loops?*
(4) Let us fix a particular loop γ^* which surrounds $(0,0)$. Show that

$$\frac{\sum_{\sigma:\gamma^*\in\mathcal{C}(\sigma)}\prod_{\gamma\in\mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|}}{\sum_\sigma\prod_{\gamma\in\mathcal{C}(\sigma)} e^{-2\beta|\gamma|}} \leq 1.$$

(5) Show that $\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right)$ is bounded above by $\sum_{\ell\geq 1}\ell 4^\ell e^{-2\beta\ell}$.
(6) Conclude that there exists $0 < \beta < \infty$ (large enough) such that

$$\liminf_{\delta\to 0} \mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99.$$