Let $\Omega \subseteq \mathbb{Z}^2$ where Ω is a bounded, connected open subset of the plane. Let $\mathbb{V}$ denote the vertices and $\mathbb{E}$ the edges of Ω.

Exercise 1. *High-temperature expansion and positive correlations*

(1) Recall the high-temperature expansion of the Ising model. Concretely, describe $Z^{\emptyset}_{\Omega,\beta}$ and $\mathbb{E}^{\emptyset}_{\Omega,\beta}[\sigma_x\sigma_y]$ for $x, y \in \mathbb{V}$.

Solution. The high-temperature expansion for the partition function and the insertion of two spins is:

$$\begin{aligned} Z^{\emptyset}_{\Omega,\beta} = \sum_{\sigma} e^{-\beta\mathcal{H}(\sigma)} &= (\cosh\beta)^{|\mathbb{E}|}\, 2^{|\mathbb{V}|} \sum_{\mathcal{E}\in\mathcal{C}} \tanh(\beta)^{|\mathcal{E}|} \\ \mathbb{E}^{\emptyset}_{\Omega,\beta}[\sigma_x\sigma_y] = \sum_{\sigma} \sigma_x\sigma_y e^{-\beta\mathcal{H}(\sigma)} &= (\cosh\beta)^{|\mathbb{E}|}\, 2^{|\mathbb{V}|} \sum_{\mathcal{E}\in\mathcal{C}_{x,y}} \tanh(\beta)^{|\mathcal{E}|} \end{aligned}$$

where $\mathcal{C}$ is the collection of edge sets such that every vertex is incident to an even number of edges and $\mathcal{C}_{x,y}$ is the collection of edge sets where every vertex is incident to an even number of edges except for x and y which are incident to an odd number of edges (i.e. $\mathcal{C}$ is the set of set of loops and $\mathcal{C}_{x,y}$ is the set of loops with a path $x \leftrightarrow y$) .

(2) Show that for any inverse temperature $\beta \in (0, \infty)$, we have

$$\forall x, y \in \mathbb{V},\ \mathbb{E}^{\emptyset}_{\Omega,\beta}[\sigma_x\sigma_y] > 0.$$

Solution. We have that $\mathbb{E}^{\emptyset}_{\Omega,\beta}[\sigma_x\sigma_y] = \frac{\sum \sigma_x\sigma_y e^{-\beta\mathcal{H}(\sigma)}}{\sum_\sigma e^{-\beta\mathcal{H}(\sigma)}} = \frac{\sum_{\mathcal{E}\in\mathcal{C}_{x,y}} \tanh(\beta)^{|\mathcal{E}|}}{\sum_{\mathcal{E}\in\mathcal{C}} \tanh(\beta)^{|\mathcal{E}|}} > 0$ since every term in $\sum_{\mathcal{E}\in\mathcal{C}_{x,y}} \tanh(\beta)^{|\mathcal{E}|}$ and $\sum_{\mathcal{E}\in\mathcal{C}} \tanh(\beta)^{|\mathcal{E}|}$ is positive and the sum in the numerator is non-empty.

Exercise 2. *Kramers-Wannier duality*

Consider the Ising model on Ω with free boundary conditions, at the self-dual inverse temperature $\beta_c = \frac{1}{2}\ln\left(1+\sqrt{2}\right)$. Fix two neighbouring vertices $x, y \in \mathbb{V}$ connected by the edge $e = \{x, y\} \in \mathbb{E}$. Write $\mathcal{C} \subseteq 2^E$ for the collection of subsets $\mathcal{E} \subseteq \mathbb{E}$ such that every vertex is incident to an even (possibly zero) number of edges in $\mathcal{E}$ (informally, $\mathcal{E}$ is a set of loops formed by elements of $\mathbb{E}$). Similarly, write $\mathcal{C}_{x,y}$ for the collection of $\mathcal{E}_{x,y} \subseteq \mathbb{E}$ such that every vertex except for x, y is incident to an even number of edges in $\mathcal{E}_{x,y}$, while x and y are both incident to an odd number of edges in $\mathcal{E}_{x,y}$. Write

$$Z(\mathcal{C}) = \sum_{\mathcal{E}\in\mathcal{C}} \exp\left(-2\beta_c\,|\mathcal{E}|\right) = \sum_{\mathcal{E}\in\mathcal{C}} (\tanh\beta_c)^{|\mathcal{E}|}, \quad Z(\mathcal{C}_{x,y}) = \sum_{\mathcal{E}_{x,y}\in\mathcal{C}_{x,y}} \exp\left(-2\beta_c\,|\mathcal{E}_{x,y}|\right).$$

(1) Express the spin correlation $\mathbb{E}^{\emptyset}_{\Omega,\beta_c}[\sigma_x\sigma_y]$ of two neighbouring vertices x, y in terms of $Z(\mathcal{C})$ and $Z(\mathcal{C}_{x,y})$.

Solution. The spin correlation $\mathbb{E}^{\emptyset}_{\Omega,\beta_c}[\sigma_x\sigma_y] = \frac{\sum_\sigma \sigma_x\sigma_y e^{-\beta\mathcal{H}(\sigma)}}{\sum_\sigma e^{-\beta\mathcal{H}(\sigma)}}$, and using the high-temperature expansion this gives:

$$\mathbb{E}^{\emptyset}_{\Omega,\beta_c}[\sigma_x\sigma_y] = \frac{Z(\mathcal{C}_{x,y})}{Z(\mathcal{C})}.$$

(2) Recall Kramers-Wannier duality.

Solution. The Kramers-Wannier duality can be formulated starting from an Ising model with free boundary conditions at arbitrary inverse temperature β on the graph $\Omega = (\mathbb{V}, \mathbb{E})$. The duality consists in obtaining an Ising model with $+$ boundary conditions at inverse temperature β^* on the dual graph $\Omega^* = (\mathbb{V}^*, \mathbb{E}^*)$, where

(a) the parameter β^* satisfies the relations

$$\exp(-2\beta^*) = \tanh(\beta) \quad \Longleftrightarrow \quad \sinh(\beta^*)\sinh(\beta) = 1,$$

which in particular implies that $\beta^*(\beta)$ is decreasing in β and $\beta^*(\beta) = \beta \quad \Longleftrightarrow \quad \beta = \beta_c := \frac{1}{2}\ln(\sqrt{2}+1)$,

(b) $\mathbb{V}^*$ is the set of faces of Ω and $\mathbb{E}^*$ is the set of dual edges (pairs of faces sharing an edge in $\mathbb{E}$).

Using the high-temperature expansion of the partition function $Z^{\emptyset}_{\Omega,\beta}$ of the Ising model on $\Omega = (\mathbb{V}, \mathbb{E})$ and the low-temperature expansion of the partition function $Z^{+}_{\Omega^*,\beta^*}$ on $\Omega^* = (\mathbb{V}^*, \mathbb{E}^*)$ we get:

$$\begin{aligned}
\text{low-temperature expansion of } Z^+_{\Omega^*,\beta^*}: \quad Z^+_{\Omega^*,\beta^*} &= e^{\beta^*|\mathbb{E}^*|} \sum_{\sigma\in\{\pm1\}^{\Omega^*}} \prod_{\gamma\in\mathcal{C}(\sigma)} e^{-2\beta^*|\gamma|} \\
\text{high-temperature expansion of } Z^\emptyset_{\Omega,\beta}: \quad Z^\emptyset_{\Omega,\beta} &= \cosh(\beta)^{|\mathbb{E}|} 2^{|\mathbb{V}|} \sum_{\mathcal{E}\in\mathcal{C}} \tanh(\beta)^{|\mathcal{E}|}
\end{aligned}$$

where $\mathcal{C}(\sigma)$ is the set of loops in Ω defined by the edges connecting spins of σ with different values and where $\mathcal{C}$ is the collection of sets of loops in Ω. Thus, $\{\mathcal{C}(\sigma) \mid \sigma \in \pm 1^{\Omega^*}\} = \{\mathcal{E} \mid \mathcal{E} \in \mathcal{C}\}$ since both sides describe the collection of sets of loops on Ω. We can rewrite the high-temperature expansion as $Z^\emptyset_{\Omega,\beta} = (\cosh\beta)^{|\mathbb{E}|} 2^{|\mathbb{V}|} \sum_{\sigma\in\{\pm1\}^{\Omega^*}} \prod_{\gamma\in\mathcal{C}(\sigma)} \tanh(\beta)^{|\gamma|}$.

For β satisfying $tanh(\beta) = e^{-2\beta^*|\gamma|}$ we thus get the following expressions:

$$\begin{aligned}
\text{low-temperature expansion of } Z^+_{\Omega^*,\beta^*}: \quad Z^+_{\Omega^*,\beta^*} &= e^{\beta^*|\mathbb{E}^*|} \sum_{\sigma\in\{\pm1\}^{\Omega^*}} \prod_{\gamma\in\mathcal{C}(\sigma)} e^{-2\beta^*|\gamma|} \\
\text{high-temperature expansion of } Z^\emptyset_{\Omega,\beta}: \quad Z^\emptyset_{\Omega,\beta} &= \cosh(\beta)^{|\mathbb{E}|} 2^{|\mathbb{V}|} \sum_{\sigma\in\{\pm1\}^{\Omega^*}} \prod_{\gamma\in\mathcal{C}(\sigma)} e^{-2\beta^*|\gamma|}.
\end{aligned}$$

Hence we have that

$$e^{-\beta^*|\mathbb{E}^*|} Z^+_{\Omega^*,\beta^*} = \cosh(\beta)^{-|\mathbb{E}|} 2^{-|\mathbb{V}|} Z^\emptyset_{\Omega,\beta}.$$

Remark. Assuming that Ω is a "reasonable domain" such as a large enough square, we have that $|\mathbb{E}^*| \approx |\mathbb{E}| \approx 2|\mathbb{V}|$, and after a bit of rewriting we obtain: $\tanh(\beta)^{|\mathbb{V}|} \cosh(\beta)^{|\mathbb{V}|} 2^{|\mathbb{V}|} Z^+_{\Omega^*,\beta^*} \approx Z_{\Omega,\beta}$. For the critical inverse temperature β_c it holds: $\tanh(\beta_c)\cosh(\beta_c)2 = 1$. Thus, the dual model has the same partition function as the original model. One can further show (though it is not obvious at this point at all) that for the critical inverse temperature β_c the two limiting models are essentially the same, and have the same limiting probability distribution.

(3) Now, write $\mathcal{C} = \mathcal{C}^e \cup \mathcal{C}^{-e}$ where $\mathcal{C}^e$ is the collection of $\mathcal{E} \in \mathcal{C}$ with $e \in \mathcal{E}$ and $\mathcal{C}^{-e} = \mathcal{C} \setminus \mathcal{C}^e$. Decompose the sum $Z = Z(\mathcal{C}^{-e}) + Z(\mathcal{C}^e)$. By Kramers-Wannier duality, we have a dual Ising model on the *faces* of the lattice with $+$ boundary conditions. Suppose the two faces separated by e are denoted f_1, f_2. Recall the low temperature expansion: what are the probabilities

$$\mathbb{P}^+_{\Omega^*,\beta_c}[\sigma_{f_1} = \sigma_{f_2}], \ \mathbb{P}^+_{\Omega^*,\beta_c}[\sigma_{f_1} \neq \sigma_{f_2}]$$

in terms of $Z(\mathcal{C}),\ Z(\mathcal{C}^e),\ Z(\mathcal{C}^{-e})$? What is $\mathbb{E}^+_{\Omega^*,\beta_c}[\sigma_{f_1}\sigma_{f_2}]$?

Solution. In the low-temperature expansion, we put an edge between two spins if and only if they disagree. Thus, we must put an edge between f_1 and f_2 if and only if $\sigma_{f_1} \neq \sigma_{f_2}$. Thus $\mathbb{P}^+_{\Omega^*,\beta_c}[\sigma_{f_1} = \sigma_{f_2}] = \frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})}$ and $\mathbb{P}^+_{\Omega^*,\beta_c}[\sigma_{f_1} \neq \sigma_{f_2}] = \frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})}$. Now the value of $\mathbb{E}^+_{\Omega^*,\beta_c}[\sigma_{f_1}\sigma_{f_2}]$ is given by

$$\mathbb{E}^+_{\Omega^*,\beta_c}[\sigma_{f_1}\sigma_{f_2}] = \mathbb{P}^+_{\Omega^*,\beta_c}[\sigma_{f_1} = \sigma_{f_2}] - \mathbb{P}^+_{\Omega^*,\beta_c}[\sigma_{f_1} \neq \sigma_{f_2}] = \frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})} - \frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})}.$$

(4) Note that there is a bijection from $\mathcal{C}$ to $\mathcal{C}_{x,y}$: given $\mathcal{E} \in \mathcal{C}^e$, $\mathcal{E}\setminus\{e\} \in \mathcal{C}_{x,y}$, and given $\mathcal{E} \in \mathcal{C}^{-e}$, $\mathcal{E}\cup\{e\} \in \mathcal{C}_{x,y}$. This also means there is a one-to-one correspondence between the terms of $Z(\mathcal{C}) = Z(\mathcal{C}^e) + Z(\mathcal{C}^{-e})$ and $Z(\mathcal{C}_{x,y})$. Express $Z(\mathcal{C}_{x,y})$ in terms of $Z(\mathcal{C}^e)$ and $Z(\mathcal{C}^{-e})$.

Solution. We have

$$\begin{aligned}
Z(\mathcal{C}_{x,y}) &= \sum_{\mathcal{E}_{x,y}\in\mathcal{C}_{x,y}} \exp(-2\beta_c|\mathcal{E}_{x,y}|) \\
&= \sum_{\mathcal{E}\in\mathcal{C}^{-e}} \exp(-2\beta_c|\mathcal{E}\cup\{e\}|) + \sum_{\mathcal{E}\in\mathcal{C}^e} \exp(-2\beta_c|\mathcal{E}\setminus\{e\}|) \\
&= \sum_{\mathcal{E}\in\mathcal{C}^{-e}} e^{-2\beta_c} \exp(-2\beta_c|\mathcal{E}|) + \sum_{\mathcal{E}\in\mathcal{C}^e} e^{+2\beta_c} \exp(-2\beta_c|\mathcal{E}|)
\end{aligned}$$

which gives

$$Z(\mathcal{C}_{x,y}) = e^{2\beta_c} Z(\mathcal{C}^e) + e^{-2\beta_c} Z(\mathcal{C}^{-e}).$$

(5) Assuming that as we take progressively larger $\Omega \subseteq \mathbb{Z}^2$, $\mathbb{E}^\emptyset_{\Omega,\beta_c}(\sigma_x\sigma_y)$ and $\mathbb{E}^+_{\Omega^*,\beta_c}(\sigma_{f_1}\sigma_{f_2})$ both tend to a single positive number μ, compute μ by using the above results.

Solution. We showed that

$$\mathbb{E}^{\emptyset}_{\Omega,\beta_c}[\sigma_x\sigma_y] = \frac{Z(\mathcal{C}_{x,y})}{Z(\mathcal{C})} = e^{2\beta_c}\frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})} + e^{-2\beta_c}\frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})}$$

and

$$\mathbb{E}^{+}_{\Omega^*,\beta_c}[\sigma_{f_1}\sigma_{f_2}] = \frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})} - \frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})}.$$

Let us suppose that $\mathbb{E}^{\emptyset}_{\Omega,\beta_c}(\sigma_x\sigma_y) = \mathbb{E}^{+}_{\Omega^*,\beta_c}(\sigma_{f_1}\sigma_{f_2})$. We get the equation

$$\frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})} - \frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})} = e^{2\beta_c}\frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})} + e^{-2\beta_c}\frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})}.$$

Let us remark that we also have

$$\frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})} + \frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})} = 1.$$

This results in a system of two equations with two unknowns, which solution is $\frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})} = \frac{2+\sqrt{2}}{4}$ and $\frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})} = \frac{2-\sqrt{2}}{4}$. Hence we obtain

$$\mu = \frac{Z(\mathcal{C}^{-e})}{Z(\mathcal{C})} - \frac{Z(\mathcal{C}^e)}{Z(\mathcal{C})} = \frac{\sqrt{2}}{2}.$$

Exercise 3. *$\beta \to \infty$ and boundary conditions*

Consider the Ising model on the lattice $\mathbb{Z}^2 \cap [0,N]^2$ with $+$ spins on the boundary vertices $\{-1\}\times[0,N]\cup[0,N]\times\{N+1\}$ and $-$ spins on the boundary vertices $\{N+1\}\times[0,N]\cup[0,N]\times\{-1\}$. Describe the $\beta\to\infty$ limit of the model.

Hint: in a previous exercises sheet, you already studied the $\beta \to \infty$ limit of an Ising model with free boundary conditions. What is the limiting distribution? Use the low temperature expansion to study the limit, and use a combinatorial argument to count the number of configurations which have a non-zero probability.

Solution. We have already seen that the $\beta \to \infty$ limit of this model is given by the uniform measure on the configurations with lowest energy (Exercise 2 Sheet 11). We just have to understand what are these configurations and how many they are. In order to do so, we consider the low temperature expansion: we draw edges separating opposite spins and we get a representation of the spin configurations as edge sets $\mathcal{E} \in 2^{E^*}$ (where E^* is the set of edges in the dual). The energy is given by $2|\mathcal{E}|$, and for any configuration, we see a path from the left-bottom corner to the right-top corner and some loops. For any lowest energy configuration there will not be a loop since it would only add more energy. Thus we have the following combinatorial problem: if we have an $N\times N$ square, how many shortest length edge-paths are there from the left-bottom corner to the right-top corner? Starting from the left-bottom corner, we either go up or right by one edge: we need to do N up moves and N right moves. So there are $\binom{2N}{N}$ such paths; a corresponding spin configuration has plus spins above the path and minus below.

As $\beta\to\infty$, Ising probabilities are uniformly distributed across the $\binom{2N}{N}$ such configurations.