Let $\Omega \subseteq \mathbb{Z}^2$ be the discretization of a bounded, connected open subset of the plane. Let $\mathbb{V}$ denote the vertices and $\mathbb{E}$ the edges of Ω.

Exercise 1. *High-temperature expansion and positive correlations*

Consider the Ising model on Ω with free boundary conditions and inverse temperature β.

(1) Recall the high-temperature expansion of the Ising model. Concretely, describe $Z^{\emptyset}_{\Omega,\beta}$ and $\mathbb{E}^{\emptyset}_{\Omega,\beta}[\sigma_x\sigma_y]$ for $x, y \in \mathbb{V}$.

(2) Show that for any inverse temperature $\beta \in (0,\infty)$, we have

$$\forall x, y \in \mathbb{V},\ \mathbb{E}^{\emptyset}_{\Omega,\beta}[\sigma_x\sigma_y] > 0.$$

Exercise 2. *Kramers-Wannier duality*

Consider the Ising model on Ω with free boundary conditions, at the self-dual inverse temperature $\beta_c = \frac{1}{2}\ln\left(1+\sqrt{2}\right)$. Fix two neighbouring vertices $x, y \in \mathbb{V}$ connected by the edge $e = \{x, y\} \in \mathbb{E}$. Write $\mathcal{C} \subseteq 2^E$ for the collection of subsets $\mathcal{E} \subseteq \mathbb{E}$ such that every vertex is incident to an even (possibly zero) number of edges in $\mathcal{E}$ (informally, $\mathcal{E}$ is a set of loops formed by elements of $\mathbb{E}$). Similarly, write $\mathcal{C}_{x,y}$ for the collection of $\mathcal{E}_{x,y}$ such that every vertex except for x, y is incident to an even number of edges in $\mathcal{E}_{x,y}$, while x and y are both incident to an odd number of edges in $\mathcal{E}_{x,y}$. Write

$$Z(\mathcal{C}) = \sum_{\mathcal{E}\in\mathcal{C}} \exp\left(-2\beta_c\left|\mathcal{E}\right|\right) = \sum_{\mathcal{E}\in\mathcal{C}} \left(\tanh\beta_c\right)^{|\mathcal{E}|},\quad Z(\mathcal{C}_{x,y}) = \sum_{\mathcal{E}_{x,y}\in\mathcal{C}_{x,y}} \exp\left(-2\beta_c\left|\mathcal{E}_{x,y}\right|\right).$$

(1) Express the spin correlation $\mathbb{E}^{\emptyset}_{\Omega,\beta_c}[\sigma_x\sigma_y]$ of two neighbouring vertices x, y in terms of $Z(\mathcal{C})$ and $Z(\mathcal{C}_{x,y})$.

(2) Recall Kramers-Wannier duality.

(3) Now, write $\mathcal{C} = \mathcal{C}^e \cup \mathcal{C}^{-e}$ where $\mathcal{C}^e$ is the collection of $\mathcal{E} \in \mathcal{C}$ with $e \in \mathcal{E}$ and $\mathcal{C}^{-e} = \mathcal{C} \setminus \mathcal{C}^e$. Decompose the sum $Z = Z(\mathcal{C}^{-e}) + Z(\mathcal{C}^e)$. By Kramers-Wannier duality, we have a dual Ising model on the *faces* of the lattice with $+$ boundary conditions. Suppose the two faces separated by e are denoted f_1,f_2. Recall the low temperature expansion: what are the probabilities

$$\mathbb{P}^{+}_{\Omega^*,\beta_c}[\sigma_{f_1} = \sigma_{f_2}],\ \mathbb{P}^{+}_{\Omega^*,\beta_c}[\sigma_{f_1} \neq \sigma_{f_2}]$$

in terms of $Z(\mathcal{C})$, $Z(\mathcal{C}^e)$, $Z(\mathcal{C}^{-e})$? What is $\mathbb{E}^{+}_{\Omega^*,\beta_c}[\sigma_{f_1}\sigma_{f_2}]$?

(4) Note that there is a bijection from $\mathcal{C}$ to $\mathcal{C}_{x,y}$: given $\mathcal{E} \in \mathcal{C}^e$, $\mathcal{E} \setminus \{e\} \in \mathcal{C}_{x,y}$, and given $\mathcal{E} \in \mathcal{C}^{-e}$, $\mathcal{E} \cup \{e\} \in \mathcal{C}_{x,y}$. This also means there is a one-to-one correspondence between the terms of $Z(\mathcal{C}) = Z(\mathcal{C}^e) + Z(\mathcal{C}^{-e})$ and $Z(\mathcal{C}_{x,y})$. Express $Z(\mathcal{C}_{x,y})$ in terms of $Z(\mathcal{C}^e)$ and $Z(\mathcal{C}^{-e})$.

(5) Assuming that as we take progressively larger $\Omega \subseteq \mathbb{Z}^2$, $\mathbb{E}^{\emptyset}_{\Omega,\beta_c}(\sigma_x\sigma_y)$ and $\mathbb{E}^{+}_{\Omega^*,\beta_c}(\sigma_{f_1}\sigma_{f_2})$ both tend to a single positive number μ, compute μ by using the above results.

Exercise 3. $\beta \to \infty$ *and boundary conditions*

Consider the Ising model on the lattice $\mathbb{Z}^2 \cap [0, N]^2$ with $+$ spins on the boundary vertices $\{-1\} \times [0, N] \cup [0, N] \times \{N + 1\}$ and $-$ spins on the boundary vertices $\{N + 1\} \times [0, N] \cup [0, N] \times \{-1\}$. Describe the $\beta \to \infty$ limit of the model.

Hint: in a previous exercises sheet, you already studied the $\beta \to \infty$ limit of an Ising model with free boundary conditions. What is the limiting distribution? Use the low temperature expansion to study the limit, and use a combinatorial argument to count the number of configurations which have a non-zero probability.