Exercise 1. *Dimers on the $2 \times k$ checker-board*

(1) Show that the number N_k of dimer configurations (complete matchings) on a checker-board of size $2 \times k$ satisfies the Fibonacci relation: $N_k + N_{k+1} = N_{k+2}$. What are N_1, N_2?

Solution. Clearly $N_1 = 1$ and $N_2 = 2$. Let us consider a $2 \times (k + 2)$ checkerboard domino tiling. Either:

(a) The last column (the $(k + 2)$-th one) is tiled by a single vertical domino, then the remaining must be tiled by dominos : the $2 \times (k + 1)$ checkerboard formed by erasing the last column is dimer tiled: they areN_{k+1} possibilities.

(b) The last column is not tiled by a single vertical domino, but by two horizontal dominos: this means that there are N_k other possibilities for the remaining $2 \times k$ checker-board. Thus $N_{k+2} = N_{k+1} + N_k$.

(2) Write down N_k using the Kasteleyn determinant: this way we obtain a non-trivial trigonometric recurrence relation.

Solution. The Kasteleyn determinant gives

$$N_k = \sqrt{\prod_{j=1}^{2} \prod_{j'=1}^{k} \left(2\cos\left(\frac{\pi j}{3}\right) + 2i\cos\left(\frac{\pi j'}{k+1}\right)\right)}.$$

Exercise 2. *3d Ising Model and dimer configurations*

Consider a 3-dimensional Ising model on a cube divided into N^3 equal size cube spin cells (i.e., each cube is assigned a spin value). For convenience, we will call each cube's six sides top-bottom-left-right-far-near, with opposite sides labelled by opposite words. We impose plus boundary conditions on the top-right-near sides, and minus on the bottom-left-far sides. We would like to study the zero temperature limit $\beta \to \infty$, namely to study the configurations that have non-zero probability in the limit $\beta \to \infty$.

(1) Recall the $2D$ version of this problem from last week's exercise sheet. What is the analogue of the interface between two corners in the $3D$ case? *Hint: If you are having a hard time precisely formulating the surface, describe it as a surface of a 3-dimensional shape, formed by stacking little cubes.*

Solution. The interface is the surface between the plus and minus cubic cells, and its surface area needs to be the minimum of such surface areas. In such an Ising configuration, a minus spin should always be adjacent to a minus spin to its bottom, left, and far sides; i.e. if minus spins are stacked as smaller cubes (plus spins being vacant) within the bigger cubic lattice, the stacking needs to be stable when we rotate it and set it with its bottom, left, or far sides down. Otherwise, sliding down the minus cube without a support will reduce surface area.

Equivalently, we can require any 2-dimensional cross section of the configuration parallel to the six sides to be a zero temperature 2D Ising configuration as in last week's exercise.

(2) Lowest energy configurations in the $3D$ Ising model can be mapped one-to-one to hexagonal lattice dimer configurations (with N hexagonal faces on each side; we place dimers on the edges of this lattice to cover all the vertices). To see this, interpret the following picture of Figure 0.1 once as a $3D$ interface between plus and minus spins, and then as a hexagonal lattice dimer tiling (first find the hexagonal lattice hidden in the picture!). Describe how we could draw the Ising interface given a hexagonal lattice dimer tiling.

Solution. Consider a triangular lattice which is tiled using shapes consisting of two triangles connected together by one edge. It is easy to see that this is equivalent both to the 3D Ising model 0 temperature minimal configurations and to hexagonal lattice dimer configurations - this correspondence should be apparent from Figure 0.2.

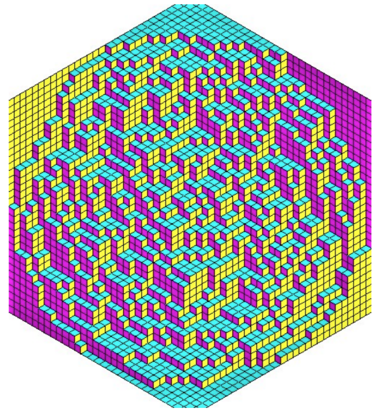

Figure 0.1. Richard Kenyon's illustration of a random tiling of a regular hexagon (https://gauss.math.yale.edu/~rwk25/gallery/bppsim.gif)

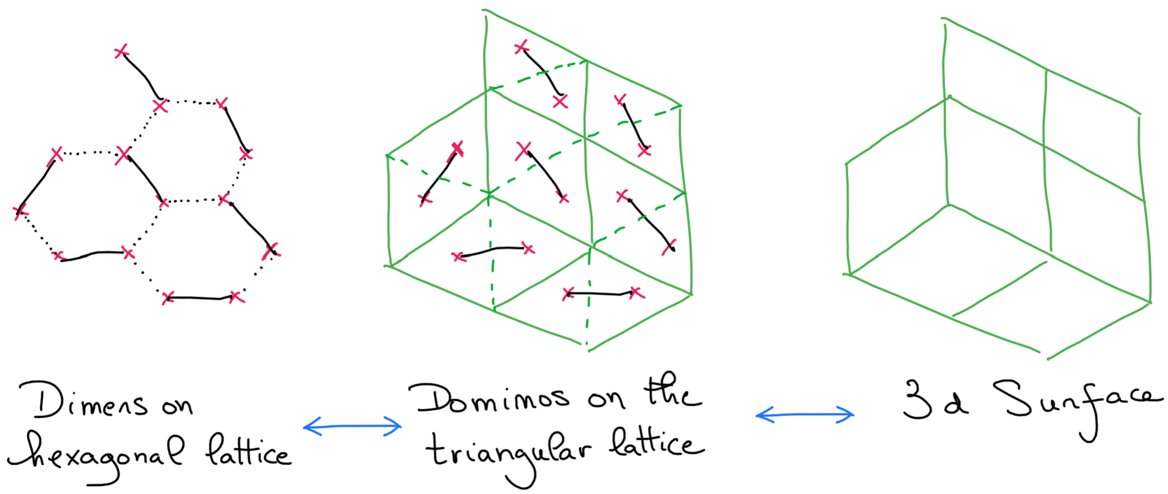

Figure 0.2. Relation between dimers, dominos and cube stackings.