Let $A \subseteq \mathbb{Z}^d$ be a finite domain. Recall that the Green's function on A is the function which for each $y \in A$ and each $x \in A \cup \partial A$ takes the value:

$$G_A(x,y) = \mathbb{E}\left[\#\left\{0 \le n < \tau_A : S_n^x = y\right\}\right]$$

where $(S_n^x)_n$ is the simple random walk started at x and $\tau_A = \min\{n \ge 0 : S_n^x \in \partial A\}$.

We also recall that the Harmonic measure on A associated to a subset $B \subseteq \partial A$ is the function which for each $x \in A \cup \partial A$ takes the value:

$$H_A(x,B) = \mathbb{P}\left[S_{\tau_A}^x \in B\right].$$

Exercise 1. *General knowledge*

(1) For $y \in A$, recall what is the discrete PDE satisfied by the function $f(x) = G_A(x,y)$.
(2) For $y \in \partial A$, recall what is the discrete PDE satisfied by the harmonic measure $f(x) = H_A(x,\{y\})$.
(3) In this question, a *salary* is a function $s : A \to \mathbb{R}$ and an *exit bonus* is a function $b : \partial A \to \mathbb{R}$. Given a path $\omega = (\omega_0, \dots, \omega_n)$ in $A \cup \partial A$ such that only $\omega_n \in \partial A$, the *reward* associated with ω is $r_{s,b} = \sum_{k=0}^{n-1} s(\omega_k) + b(s_n)$. Give an interpretation of $G_A(x,y)$ and $H_A(x,\{y\})$ as an *expected reward.*
(4) Give an explicit solution to

$$\begin{cases} \Delta f = 0 & \text{in } A \\ f = F & \text{in } \partial A \end{cases} \tag{0.1}$$

in terms of $\{H_A(x,\{y\})\}_{x,y}$, and give a interpretation of the solution as an *expected reward.*
(5) Solve

$$\begin{cases} \Delta f = \rho & \text{in } A \\ f = 0 & \text{in } \partial A \end{cases} \tag{0.2}$$

in terms of the Green's function and give an interpretation of $f(x)$ as an *expected reward.*
(6) Explain why

$$G_A(x,x) = \sum_{\omega : x \to x, \omega \subset A} \left(\frac{1}{2d}\right)^{|\omega|}$$

where $|\omega|$ is the length of the path $\omega = \left[x = \omega_0, \cdots, \omega_{|\omega|} = x\right]$.

Exercise 2. *Discretisation of PDEs : the equilibrium case*

We want to study the discrete PDEs :

$$\begin{cases} \Delta f = \rho & \text{in } A \\ f = F & \text{in } \partial A \end{cases} \tag{0.3}$$

and to give an explicit formulation in terms of the given functions ρ, F, the Green's function G_A and the harmonic measure $H_A(x,y)$

(1) Recall why there is at most one solution to the system (0.3).
(2) Solve the system (0.3) and give an interpretation of $f(x)$ as an *expected reward.*

Exercise 3. *Discretisation of PDEs: the evolution case*

We want to give an explicit formulation and a probabilistic interpretation of the solution to the discrete partial differential equation:

$$\begin{cases} f(x,t+1) - f(x,t) = \Delta f(x,t) & \text{for } (x,t) \in A \times \mathbb{N} \\ f(x,t) = F(x) & \text{for } (x,t) \in \partial A \times \mathbb{N} \cup A \times \{0\} \end{cases} \tag{0.4}$$

where $f : (A \cup \partial A) \times \mathbb{N} \to \mathbb{R}$.

(1) Prove that the solution to (0.4) is unique.
(2) Suppose that $f(\cdot,t)$ converges to a function $g(\cdot)$ when t goes to infinity. What discrete partial differential equation does g satisfy? Thus, which function (or modification of it) should appear in the explicit formulation: the Harmonic measure or the Green's function?

(3) Write the discrete PDE as $\Delta_t f(x,t) = 0$ where Δ_t is a linear operator.
(4) Find an explicit formulation of a solution. *Hint: For $t \in \mathbb{N}$ consider the random variable $S_{\tau_A \wedge t}$* where $\tau_A \wedge t = min\{\tau_A, t\}$ and take its expected value under the image of F.
(5) Let us consider the oriented graph $A^{\rightarrow} = A \times \mathbb{N} \subseteq \mathbb{Z}^{d+1}$ with neighbours of the form $(x_1, t_1) \rightsquigarrow (x_2, t_2)$ if and only if $x_1 \sim x_2$ in A and $t_2 = t_1 + 1$ ($\rightsquigarrow$ represents the oriented edge pointing from (x_1, t_1) to (x_2, t_2)). We define the Laplacian on $A^{\rightarrow}$ for a function $f : A \cup \partial A^{\rightarrow} \to \mathbb{R}$ as

$$\Delta f(\bar{x}) = \frac{1}{\#\{\bar{y} \rightsquigarrow \bar{x}\}} \sum_{\bar{y} \rightsquigarrow \bar{x}} \left(f(\bar{y}) - f(\bar{x})\right).$$

   (a) What is $\partial A^{\rightarrow}$?
   (b) Show that f is a solution to (0.4) if and only if f is harmonic on $A^{\rightarrow}$ with suitable boundary conditions.
   (c) Show that the harmonic measure $H_{A^{\rightarrow}}((x,t), \{(y,s)\})$ is equal to

$$\begin{cases} \mathbb{P}^x\left(S_{\tau_A} = y \text{ and } \tau_A = t - s\right) & \text{if } s > 0 \\ \mathbb{P}^x\left(S_t = y \text{ and } \tau_A \geq t\right) & \text{if } s = 0 \end{cases}$$

   where we recall that $(S_n^x)_{n=0}^{\infty}$ is the simple random walk on A starting at x.
   (d) Using the last question, give the explicit formulation of (0.4).

Exercise 4. *Discretisation of PDEs: the time-dependent boundary condition.*

We want to give an explicit formulation and a probabilistic interpretation of the solution to the discrete partial differential equation:

$$\begin{cases} \Delta f(x,t) = f(x,t+1) - f(x,t) & \text{for } (x,t) \in A \times \mathbb{N} \\ f(x,t) = F(x,t) & \text{for } (x,t) \in \partial A \times \mathbb{N} \cup A \times \{0\} \end{cases}$$

where $f : A \cup \partial A \to \mathbb{R}$.

Following the same ideas used for the point 4. of Exercise 3, give an explicit formulation and a probabilistic interpretation of the solution to the latter discrete partial differential equation.