**Exercise 1.** *Neumann boundary conditions and Dirichlet boundary conditions*

Let us consider the rectangle $A = [|1, n|] \times [|1, m|]$ in $\mathbb{Z}^2$, and its boundary $\partial A$ which is the set of vertices in $\mathbb{Z}^2 \setminus A$ adjacent to a vertex of $A$. We define the normal derivative at $y \in \partial A$ as :

$$\partial_n f(y) = f(x) - f(y),$$

where $x$ is the unique neighbour of $y$ in $A$. We denote by $\underline{\partial A}, \partial A|, \overline{\partial A}$ and $|\partial A$ the 4 parts of the boundary, respectively the lower horizontal, the right most vertical, the upper horizontal and the left-most vertical parts of $\partial A$.

(1) Prove that the problem

$$\begin{cases} \Delta f(x) = 0 & \text{in } A \\ f(y) = 0 & \text{on } \underline{\partial A} \\ f(y) = 1 & \text{on } \overline{\partial A} \\ \partial_n f(y) = 0 & \text{on } \partial A| \cup |\partial A \end{cases}$$

has a unique solution if any.

(2) Let us consider the simple random walk on $A \cup \partial A$ ($x \in A$ and $y \in \partial A$ are linked by an edge if $y$ is adjacent to $x$ in $\mathbb{Z}^2$, two points of $\partial A$ are not linked by an edge). Is $\tau_{\underline{\partial A} \cup \overline{\partial A}}$, the hitting time for $\underline{\partial A} \cup \overline{\partial A}$, finite almost surely ?

(3) Give an explicit formulation of the unique solution of the discrete PDE in terms of random walks. What is the difference with the pure Dirichlet conditions ?

*Remark.* We used a mix of conditions in order to have a finite stopping time. Yet, one can solve the Neumann problem with pure Neumann boundary conditions. In this case, the boundary conditions must satisfy some additional conditions for a solution to exist, and the solution is unique only up to a constant. This is slightly more technical. (Section 6.7 of https://www.math.uchicago.edu/~lawler/srwbook.pdf)

*Remark.* Actually we considered a rectangle for simplicity, but one can do the same with any discretisation of any domain $\Omega$ with 4 points marked on the boundary in counterclockwise order $a, b, c$ and $d$ and with Dirichlet boundary conditions on $[a, b], [c, d]$ and Neumann conditions on $[b, c], [d, e]$. Then the solution would be the imaginary part of the discretisation of the conformal mapping which sends the domain $\Omega$ to a rectangle $[0, L] \times [0, i]$ (for some $L$) which sends $a, b, c, d$ to the corners of the rectangle.

**Exercise 2.** *Green's function representation by determinant*

We consider $A \subseteq \mathbb{Z}^d$ finite. The goal of this exercise is to prove that if $x_1, \cdots, x_n \in A$ and $A_k = A \setminus \{x_1, \cdots, x_k\}$ then

$$G_A(x_1, x_1) G_{A_1}(x_2, x_2) \cdots G_{A_{n-1}}(x_n, x_n)$$

is independent of the order of $x_1, \ldots, x_n$.

*Remark.* For this exercise sheet, we will consider :

$$\Delta^+ f(x) = f(x) - \frac{1}{2d} \sum_{y \sim x} f(y)$$

($\Delta^+ = -\Delta$ is the positive definite operator).

(1) We can consider $\Delta^+$ as a linear operator $\Delta_A^+ : \mathbb{R}^A \to \mathbb{R}^A$, by considering a vector on $A$ as a function on $A \cup \partial A$ such that $f \restriction_{\partial A} = 0$. Show that :

$$G_A(x, x) = \frac{\det \Delta^+_{A \setminus \{x\}}}{\det \Delta^+_A}.$$

*Hint. Think about Cramer's rule for describing inverse matrices.*

(2) If $x_1, \cdots, x_n \in A$ and $A_k = A \setminus \{x_1, \cdots, x_k\}$, give the value of

$$G_A(x_1, x_1) G_{A_1}(x_2, x_2) \cdots G_{A_{n-1}}(x_n, x_n)$$

and prove that it is independent of the order of $x_1, \cdots, x_n$.

**Exercise 3.** *Determinant of Laplacian and uniform spanning trees : Kirchhoff's theorem*

*Remark.* If $M$ is a matrix, we denote by $M^{i,j}$ the matrix obtained by deleting the $i$-th row and $j$-column of $M$. We also denote by $M^{i,\cdot}$ the matrix obtained by deleting only the $i$-th row of $M$. The cofactor $\det^{i,j} M$ is $(-1)^{i+j} \det M^{i,j}$.

Let $G$ be a connected graph with $n$ vertices and $m$ edges (here an edge is a couple of vertices, in particular, we do not consider the case where two vertices are related by two or more edges). Recall that a spanning tree of $G$ is a connected subgraph of $G$ with no loop and which covers all vertices of $G$. Let us give a unique number between 1 and $n$ to each vertex and a unique number between 1 and $m$ to each edge. For this exercise, we denote by $\tilde{\Delta}_G$ the matrix defined by:

$$\tilde{\Delta}_G(i,j) = \delta_{i,j} \deg(i) - \delta_{j\sim i},$$

where $1 \leq i, j \leq n$ denote vertices of $G$ and $\deg(i)$ is the degree of $i$.

We will prove Kirchhoff's theorem :

$$\boxed{\text{Kirchhoff's theorem : } \#\{\text{spanning trees of G}\} = \det{}^{i,j}\left(\tilde{\Delta}_G\right)}$$

Actually, we will only show that $\#\{\text{spanning trees of G}\} = \det^{1,1}\left(\tilde{\Delta}_G\right)$, the general case can be deduced using elementary linear algebra arguments.

(1) Let the $n \times m$ incidence matrix $E$ such that the only non zero elements are given by the following: if the $k$-th edge goes between $i$ and $j$ and $i < j$ then $E_{ik} = 1$ and $E_{jk} = -1$. Show that $\tilde{\Delta}_G = EE^{\mathrm{T}}$, where $E^{\mathrm{T}}$ is the transpose of $E$.

(2) Show that $\tilde{\Delta}_G^{1,1} = E^{1,\cdot}\left(E^{1,\cdot}\right)^{\mathrm{T}}$.

(3) Prove that $m \geq n-1$ i.e. the matrix $E^{1,\cdot}$ has a horizontal shape more than a vertical shape.

(4) Recall the Cauchy-Binet formula which says that if $A$ and $B$ are two matrices of size $l \times k$ and $k \times l$ then

$$\det(AB) = \sum_{S \subset [k], \# S = l} \det\left(A_{[l],S}\right) \det\left(B_{S,[l]}\right)$$

where $[k] = \{1, \ldots, k\}$, $A_{S,[k]}$ is the matrix obtained by choosing the rows in $S$ and the columns in $[k]$. Use this formula to show that

$$\det \tilde{\Delta}_G^{1,1} = \sum_{S \subset [m], \# S = n-1} \det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right)^2.$$

(5) What represents a choice of $S \subset [m]$ in the original graph $G$ ? We want to show that $S$ forms a spanning tree of $G$ if and only if $\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = \pm 1$, and if it does not form a spanning tree then $\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = 0$.

(a) Show that if $S$ is not a spanning tree, then there exists a cycle in $S$. .

(b) Show that if $S$ is not a spanning tree then $\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = 0$.

(c) Let us suppose that $S$ is a spanning tree. Consider the vertex 1 and an edge $e$ in $S$ connected to 1 and to a vertex $i$. Prove that

$$\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = \pm \det\left(\left(E^{1,\cdot}\right)_{[n-1]\setminus\{i-1\},S\setminus\{e\}}\right).$$

(d) Conclude that if $S$ is a spanning tree, then $\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = \pm 1$.

(6) Prove Kirchhoff's theorem.

(7) How do you write the number $\#\{\text{spanning trees of G}\}$ using the Laplacian $\Delta_G$ ?