Exercise 1. *Getting familiar with dual graphs: Euler's theorem*

A graph $\mathbb{G}= (\mathbb{V}, \mathbb{E})$ is planar, if it can be embedded in the plane; i.e., if you can draw it on a plane in such a way that its edges intersect only at the endpoints. A face is any region delimited by a set of edges (usually, there is one "outer unbounded region" which is considered a face also).

For a planar graph we define its dual $\mathbb{G}'= (\mathbb{V}', \mathbb{E}')$ that has a vertex for each face of $\mathbb{G}$ and has an edge for each pair of faces in $\mathbb{G}$ that are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. Thus, each edge e of $\mathbb{G}$ has a corresponding dual edge, whose endpoints are the dual vertices corresponding to the faces on either side of e.

(1) Show that there is a natural bijection between $\mathbb{E}$ and $\mathbb{E}'$.

Solution. This bijection is already described in the definition above: each edge e of $\mathbb{G}$ has a corresponding dual edge, whose endpoints are the dual vertices corresponding to the faces on either side of e.

(2) Let T be a spanning tree of $\mathbb{G}$ and let us consider its edge complement $T^c = \mathbb{E} \setminus T$. Show that edges in the dual graph corresponding to T^c also form a spanning tree.

Solution. Let us denote by T' the subgraph in $\mathbb{G}'$ induced by T^c. Notice that if T contained a loop, this would isolate a set of faces in T'. So since T does not contain any loop, each vertex in the dual graph is in one connected component of T'. Similarly, if T' contained a loop, the loop would isolate a set of vertices of $\mathbb{G}$ and T could not be spanning the whole graph. Hence, T' is loop-less and thus, a spanning tree in the dual graph.

(3) Remember that each tree with k vertices has $k - 1$ edges.

Solution. Each tree has a leaf. By induction on the size of the tree, one obtains the statement.

(4) Using the dual graph, prove Euler's formula; i.e. let V, E, F denote the number of vertices, edges, and faces of $\mathbb{G}$, respectively; show that $V - E + F = 2$.

Solution. We pick a spanning tree T of $\mathbb{G}$ and by $E(T)$ we denote the number of edges of T. Then, $E = E(T) + E(T^c)$. Using (3) we know that $V = E(T) + 1$. Using (2), we know that T^c is a spanning tree of the dual graph, and thus $F = E(T^c) + 1$. Together, this gives that $E = V + F - 2$.

Recall that if $\mathbb{G}$ is a graph, $\mathbb{V}$ is the set of vertices of $\mathbb{G}$, and $\mathbb{E}$ is the set of edges of $\mathbb{G}$, if $p \in [0, 1]$, we can consider:

(1) *the vertex (or site) percolation*, which is a percolation on the vertices of $\mathbb{G}$. It means we consider $(X_v)_{v\in V}$ a sequence of i.i.d. Bernoulli random variables of parameter p.
(2) *the bond (or edge) percolation*, which is a percolation on the edges of $\mathbb{G}$. It means we consider $(X_e)_{e\in E}$ a sequence of i.i.d. Bernoulli random variables of parameter p.

In a site percolation, a path is composed of nearest neighbours whose labels are equal to 1, whereas in the bond percolation, a path is a concatenation of edges whose labels are equal to 1.

Exercise 2. *Probability* $\theta (p)$

Let us consider the bond percolation on $\mathbb{Z}^d$ of parameter $p \in [0, 1]$. We therefore consider i.i.d Bernoulli random variables $(X_e)_{e\in\mathbb{E}^d}$ of parameter p where $\mathbb{E}^d$ is the edge set of $\mathbb{Z}^d$. If $X_e = 1$ we say that the edge is open.

The critical probability p_c is such that if $p > p_c$, then there almost surely exists an infinite open cluster in the percolation of parameter p, and if $p < p_c$ there almost surely exists no infinite open cluster in the percolation of parameter p.

We define

$$\theta (p) = \mathbb{P}_p (0 \to \infty)$$

where by $0 \to \infty$ we mean that 0 belongs to an infinite open cluster.

(1) Show that $\mathbb{P}_p (\exists \text{ infinite open cluster}) = 0$ or 1. *Hint : Think about a general theorem which involves* 0 *and* 1.

Solution. You just need to use the Kolmogorov's zero-one law since the event $\{\exists \text{ infinite open cluster}\}$ is a tail event: even if you change the labelling (the 0 or the 1) in a finite box, you will not change the fact that there exists or not an infinite open cluster.

(2) Recall why $p \to \mathbb{P}_p (\exists \text{ infinite open cluster})$ is non-decreasing. What is the value of this function at $p = 0$ and 1? Deduce that p_c exists.

Solution. This is non-decreasing because we can use a coupling argument. We consider $(U_e)_{e\in\mathbb{E}^d}$ some $[0,1]$ uniform random variables which are independent, and we define $X_e^p = \mathbb{1}_{U_e \leq p}$ for any edge and any $p \geq 0$. Then for any p, $(X_e^p)_e$ is a percolation of parameters p and when p increases, we just add some new edges (i.e. their label changes from 0 to 1). Thus when p increases, if there was already an infinite open cluster, this cluster will remain. This implies that $p \to \mathbb{P}_p(\exists \text{ infinite open cluster})$ is non-decreasing.

When $p = 0$, there is no open edge and thus no open cluster : $\mathbb{P}_0(\exists \text{ infinite open cluster}) = 0$. When $p = 1$, there are only open edges and thus $\mathbb{Z}^d$ is an open cluster : $\mathbb{P}_1(\exists \text{ infinite open cluster}) = 1$. Thus, the function $p \to \mathbb{P}_p(\exists \text{ infinite open cluster})$ begins at 0, takes only values in $\{0,1\}$, is non-decreasing, and finishes at 0. This implies that there exists a p_c such that if $p < p_c$ then $\mathbb{P}_p(\exists \text{ infinite open cluster}) = 0$ and if $p > p_c$ then $\mathbb{P}_p(\exists \text{ infinite open cluster}) = 1$.

(3) Show that $\theta(p) > 0 \Longleftrightarrow \mathbb{P}_p(\exists \text{ infinite open cluster}) = 1$.

Solution. Let us show that $\theta(p) > 0 \Longrightarrow \mathbb{P}_p(\exists \text{ infinite open cluster}) = 1$. If $\theta(p) > 0$ this means that there is a strictly positive probability that 0 belongs to an infinite open cluster. In particular, there is a strictly positive probability that there exists an infinite open cluster. Thus $\mathbb{P}_p(\exists \text{ infinite open cluster}) > 0$. But we have seen that $\mathbb{P}_p(\exists \text{ infinite open cluster}) \in \{0,1\}$. Thus $\mathbb{P}_p(\exists \text{ infinite open cluster}) = 1$.

Let us show now that $\theta(p) > 0 \Longleftarrow \mathbb{P}_p(\exists \text{ infinite open cluster}) = 1$. Instead, let us prove the equivalent assertion that $\theta(p) = 0 \Longrightarrow \mathbb{P}_p(\exists \text{ infinite open cluster}) = 0$. If $\theta(p) = 0$, this means that the probability that 0 is in an infinite cluster is zero. By translation invariance, this implies that the probability that *any* vertex v belongs to an infinite cluster is also zero. But $\{\exists \text{ infinite open cluster}\} = \cup_v \{v \text{ belongs to an infinite open cluster}\}$. The union of events of probability zero has also a probability which is zero. This implies that $\mathbb{P}\{\exists \text{ infinite open cluster}\} = 0$.

(4) Show that $p \to \theta(p)$ is right continuous. *Hint : Use some exchange of limits.*

Solution. We consider the box $B_n = [-n,n] \times [-n,n]$, and the function $p \to \theta_n(p) = \mathbb{P}_p(0 \rightsquigarrow \partial B_n)$. We see that

(a) the function $\theta_n(p)$ is non decreasing in p (using the point 2.) and are continuous (since we are considering a finite box),

(b) at p fixed, $\theta_n(p)$ is non increasing in n (since the events are more and more difficult to achieve as n goes to infinity).

This implies that if we consider $(p_k)_k$ a decreasing sequence converging to p, we have

$$\theta(p) = \inf_n \theta_n(p) = \inf_n \inf_k \theta_n(p_k) = \inf_k \inf_n \theta_n(p_k) = \inf_k \theta(p_k)$$

where we used (in order), the point (b), the point (a), inversion of infimums (since by (a) and (b) $\theta_n(p_k)$ is non-increasing both when $n \to \infty$ and $k \to \infty$) and the point (b) at last.

(5) Draw the shape of $\theta(p)$: can we say anything for now at $p = p_c$?

Solution. By (3) θ is 0 between 0 and p_c and positive and right continuous between p_c and 1 with $\theta(1) = 1$. For now, we do not know if $\theta(p)$ jumps or not at p_c (in fact, it does not).

Remark. Similar arguments can be applied to site percolation on “gentle” infinite graphs, typically the triangle percolation.

Exercise 3. *Existence of a phase transition for $\mathbb{Z}^d$ for $d \geq 2$: $p_c \in (0,1)$.*

(1) We want to prove that $\theta(p) = 0$ when p is small enough:

(a) Show that for all $N \in \mathbb{N}^*$, $\mathbb{P}_p(0 \to \infty) \leq \mathbb{P}_p(\exists \gamma, \text{ self avoiding walk starting from 0 of length } N \text{ which is open})$.

Solution. If 0 is linked to infinity, it means that for all $N \in \mathbb{N}$ we can find a self avoiding walk starting from 0 of length N which is open (it is almost the definition of 0 linked to infinity, except that you might have considered paths of length N for any integer N, yet if you loop-erase all these paths, you get a family of self avoiding walks which size is going to infinity). Thus for all $N \in \mathbb{N}$

$$\mathbb{P}_p(0 \to \infty) \leq \mathbb{P}_p(\exists \gamma, \text{ self avoiding walk starting from 0 of length } N \text{ which is open}).$$

(b) Prove that if $p < \frac{1}{\mu_d}$ where $\mu_d \leq 2d$ is the connectivity constant of $\mathbb{Z}^d$ then $\mathbb{P}_p(0 \to \infty) = 0$. Deduce that $p_c \geq \frac{1}{\mu_d}$.

Solution. If γ is a self avoiding walk starting from 0 of length N, the probability that it is open is equal to $p^{|\gamma|}$, since $(X_e)_{e\in\gamma}$ are i.i.d. Bernoulli of parameter p. Thus, using the previous inequality, we get that for any N,

$$\begin{aligned}\mathbb{P}_p(0\to\infty) &\leq \mathbb{P}_p(\exists\gamma,\ \text{self avoiding walk starting from 0 of length } N \text{ which is open}).\\ &\leq \mu_N p^N,\end{aligned}$$

where μ_N is the number of self-avoiding walks starting from 0 of length N. Yet recall that last week, we have see that $\mu_N^{1/N} \to \mu_d$ the connectivity constant of the graph. If $p < \frac{1}{\mu_d}$ then $\lim_N (\mu_N)^{\frac{1}{N}} p < 1$ and thus there exists N such that for any $n \geq N$,

$$\left(\mu_n^{1/n}\right) p < 1-\epsilon.$$

This implies that $\mu_N p^N \to 0$ as N goes to infinity: $\mathbb{P}_p(0\to\infty) = 0$. By point (3) of the previous exercise, this also implies that $p_c \geq \frac{1}{\mu_d}$.

We still consider the bond percolation of parameter p. We have shown that p_c exists yet we have only shown that $p_c \in [0,1]$. The goal is to prove that $p_c \in (0,1)$. Recall that the connective constant (μ) of a graph was defined in the previous exercise sheet (Exercise 2).

(1) We want to prove that $\theta(p) > 0$ when p is big enough:

(a) Why do we only need to prove the case where $d = 2$?

Solution. If 0 is connected to infinity on the plane percolation with a certain probability, for higher dimensions it will be connected to infinity with a bigger probability since there is even more space to use to escape to infinity.

(b) Consider the dual graph of $\mathbb{Z}^2$: which graph is it ? Explain why a percolation on $\mathbb{Z}^2$ induces a natural percolation on the dual graph (there should be two possibilities, but one is not really interesting if you look at the next question and if you think about the crossing arguments in the square which were used in the lesson). What is the parameter of the dual percolation ?

Solution. The dual graph of $\mathbb{Z}^2$ is simply a graph obtained by translating $\mathbb{Z}^2$. Naturally, there are two possibilities to define a percolation on the dual graph using a percolation on $\mathbb{Z}^2$. Yet, one of these two possibilities is not interesting, we will see why after.
Let us consider $(X_e)_{e\in\mathbb{E}^d}$, a bond percolation on $\mathbb{Z}^2$. For any edge e, we consider the dual edge e^* which is the edge which cuts e in the dual graph. Then the two possibilities are:

$$\begin{aligned}\forall e\in\mathbb{E}^d, &\quad X_{e^*} = X_e\\ \forall e\in\mathbb{E}^d, &\quad X_{e^*} = 1-X_e\end{aligned}$$

Yet, if you consider the next question, or if you think about the RSW estimate, we would like to have a percolation on the dual lattice which tells you something when 0 is not in an infinite cluster, i.e. when there is no crossing from 0 to infinity. Thus the dual percolation has to remember the lack of edges instead of the presence of edges. This makes

$$\forall e \in \text{edges of } \mathbb{Z}^2, X_{e^*} = 1 - X_e$$

more natural.
Now the parameter of this percolation is simply $1-p$.

(c) Show that 0 is not in an infinite cluster if and only if there exists a self-avoiding cycle which surrounds 0 in the dual percolation.

Solution. If there exists a self-avoiding cycle which surrounds 0 in the dual percolation, this self-avoiding cycle disconnect 0 to the infinity : we can not go through this self-avoiding cycle (by definition of the dual percolation). Now, if 0 is not in an infinite cluster, then we can consider the cluster 0 belongs to, and we can consider the boundary of this cluster (boundary here in the dual) This boundary is the cycle we are looking for.

(d) Prove that

$$1-\theta(p) \leq C\sum_{\ell\geq 0} \ell 4^\ell (1-p)^\ell$$

where the constant C does not depend on p. *Hint : I did not choose the letter ℓ for nothing, what can it be ?*

Solution. Let us remark that $1-\theta(p)$ is the probability that 0 does not belong to an infinite cluster. We have seen that 0 does not belong to an infinite cluster if and only if there exists a self-avoiding loop which surrounds 0 in the dual percolation. Thus :

$$1-\theta(p) \leq \sum_{\gamma \text{ loops surround 0 in dual graph}} \mathbb{P}(\gamma \quad \text{open}).$$

We can split the sum into loops of length $\ell \geq 0$:

$$\sum_{\gamma \text{ loops surround 0 in dual graph}} \mathbb{P}(\gamma \quad \text{open}) = \sum_{\ell} \sum_{\gamma \text{ loops surround 0 in dual graph of length } \ell} \mathbb{P}(\gamma \quad \text{open}).$$

Let us remark that $\mathbb{P}(\gamma \quad \text{open}) = (1-p)^{\ell}$ if γ is of length ℓ since the dual percolation is a percolation of parameters $1-p$.
Thus,

$$\sum_{\ell} \sum_{\gamma \text{ sur. 0 length } \ell} \mathbb{P}(\gamma \quad \text{open}) = \sum_{\ell} (1-p)^{\ell} \# \{\text{loops of length } \ell \text{ sur. 0 in the dual graph}\}$$

We just have to find a bound on the number of loops of length ℓ which surround 0 . Yet since such a loop surrounds 0 it must cross the x axis at a point which is at a maximal distance from 0 of order ℓ (of order i.e. up to a multiplicative constant). Thus we can consider that the loop starts at a point which is at a maximal distance from 0 of order ℓ (there are ℓ such points) and then it is of length ℓ: there is still a maximum of 4^{ℓ} choices of loops once we have chosen its starting points. Thus there exists a constant $C > 0$ such that

$$\# \{\text{loops of length } \ell \text{ surrounding 0 in the dual graph}\} \leq C\ell 4^{\ell}$$

and therefore

$$1-\theta(p) \leq C \sum_{\ell \geq 0} \ell 4^{\ell} (1-p)^{\ell} = C \frac{4(1-p)}{(3-4p)^2}.$$

(e) Show that if p is big enough, $\theta(p) > 0$.

Solution. If p is big enough then $1-p$ is small enough and the r.h.s. will be smaller than 1. This means that $\theta(p) > 0$ if p is big enough.

(2) Summarize the result we got.

Solution. We proved that $p_c \in (0,1)$.

Remark. Again similar arguments can be used for site percolation on “gentle” infinite graphs, typically the triangular site percolation. One can show that for the triangle percolation, $p_c = \frac{1}{2}$. In the following exercise sheets, we will only consider the case $p_c = \frac{1}{2}$.