Exercise 1. *Duality*

(1) Show that the probability in bond percolation at $p = \frac{1}{2}$ on $\mathbb{Z}^2$ restricted to the rectangle $[0, n] \times [0; n+1]$ that there is an open crossing from top to bottom is exactly $\frac{1}{2}$.

(2) Suppose we have a discretisation of a simply connected domain Ω with smooth boundary with three distinct points $a,\ b,\ c$ on $\partial\Omega$ (in counter-clockwise order). Considering the honeycomb face percolation at $p = \frac{1}{2}$, show that the probability that there is an open cluster which connects all three boundary segments $[a, b], [b, c], [c, a]$ is $\frac{1}{2}$.

(3) Considering again the honeycomb face percolation at $p = \frac{1}{2}$ and Ω a Jordan domain with four distinct points $a_1,\ a_2,\ a_3,\ a_4$ on $\partial\Omega$ (in counter-clockwise order), prove that the two following events have the same probability (black lines represent a black connection, red lines represent a white connection). To be more precise: $\mathcal{E}_1$ is the event that there exists a black path connecting the arc a_1a_2 to the arc a_3a_4. $\mathcal{E}_2$ is the event that there exists a black path connecting the arc a_1a_2 to the arc a_3a_4 and at the same time there exists a white path connecting the black path to a_4a_1 (note that if the black path directly touches the arc a_4a_1 we still consider the event $\mathcal{E}_2$ to occur as the white path can be empty).

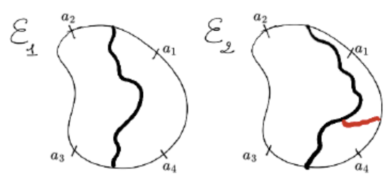
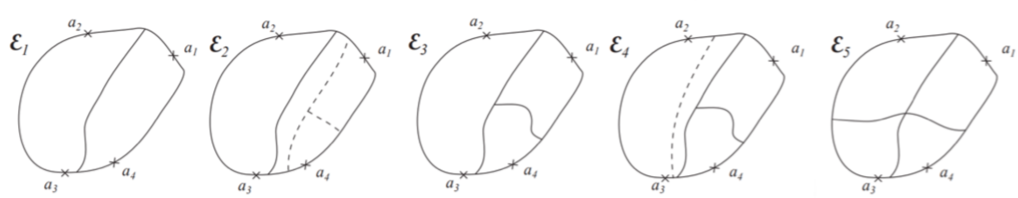


FIGURE 0.1. The events $\mathcal{E}_1$ and $\mathcal{E}_2$

(4) For the honeycomb face percolation at $p = \frac{1}{2}$, define events $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5$ as in the figure below (solid lines represent a white connection, dashed lines represent a black connection). Show that

$$\mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\mathcal{E}_4) = \frac{\mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_5)}{2} \text{ and } \mathbb{P}(\mathcal{E}_3) = \frac{\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_5)}{2}$$



*Hint 1: Prove that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$ and then $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_4 \sqcup \mathcal{E}_5$. Hint 2: For the first equality, prove the two equalities in the Figure 0.2 (*on the second equality, the black path is the right most black path)*:*

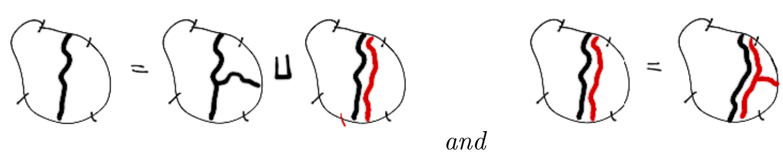


FIGURE 0.2. Steps for $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$.

Exercise 2. *Arzela-Ascoli theorem*
We will use the same notation as in the lectures:

$$\mathcal{H}_\delta^1(z) = \mathbb{P}_{\frac{1}{2}}\left(\text{a black path disconnects } \{a_1, z\} \text{ from } \{a_2, a_3\}\right)$$

where we consider the face percolation at $p = \frac{1}{2}$ on the honeycomb lattice, Ω_δ the discretization of a Jordan domain Ω, and a_1, a_2, a_3 are anti-clockwise ordered points on the boundary $\partial\Omega$ (z is a vertex of the hexagonal lattice). You have seen in the lesson how to deduce the following Hölder estimate:

$$\left|\mathcal{H}_\delta^1(x) - \mathcal{H}_\delta^1(y)\right| \leq C d_{\Omega_\delta}(x, y)^\alpha,$$

where d_{Ω_δ} is calculated by taking the length of the shortest path between x and y in Ω_δ. Our goal is to use this estimate to extract a limit for $\left(\mathcal{H}_\delta^1\right)_\delta$. For this, we extend the discrete function $\mathcal{H}_\delta^1$ defined on the vertices of Ω_δ to a continuous function defined on $\Omega \cup \partial\Omega$ by piecewise linear interpolation.

(1) Recall Arzela-Ascoli theorem.
(2) Show that $\left(\mathcal{H}_\delta^1\right)_\delta$ is uniformly bounded. *Hint: this can be done using the estimate above, but it makes more sense to go back to the definition.*
(3) Show that the above Hölder estimate implies that the family $\left(\mathcal{H}_\delta^1\right)_\delta$ is uniformly equicontinuous.
(4) Deduce that we can extract a limit in $\left(\mathcal{H}_\delta^1\right)_\delta$.